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Gravitational couplings of charged leptons in a medium

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Abstract

We calculate the leading order matter-induced corrections to the gravitational interactions of charged leptons and their antiparticles in a medium that contains electrons but not the other charged leptons, such as normal matter. The gravitational coupling, which is universal at the tree level, is found to be flavor-dependent, and also different for the corresponding antiparticles, when the corrections of $O(\alpha)$ are taken into account. General expressions are obtained for the matter-induced corrections to the gravitational mass in a generic matter background, and explicit formulas for those corrections are given in terms of the macroscopic parameters of the medium for particular conditions of the background gases.

1 Introduction

The gravitational interactions are universal in the sense that the ratio of the inertial and the gravitational masses of any particle is a constant. This fact, expressed in the form equivalence principle, is one of the basic axioms of the general theory of relativity. Although this is a feature of the theory at the classical level, it has been shown by Donoghue, Holstein and Robinett (DHR) [1, 2], that the corresponding linearized quantum theory of gravity respects this ratio, at least to $O(\alpha)$.

However, in the same series of works, it was shown that this property is lost when the particles are in the presence of a thermal background rather than the vacuum. To arrive at this idea, the inertial and the gravitational masses must be defined in the context of quantum field theory. We consider in Sec. 2 their precise definition in terms of the particle propagator and the gravitational vertex, which we will need in the subsequent work. For the moment, let us denote these two quantities by M and M' respectively and summarize the results of Refs. [1, 2]. The authors calculated the corrections for the electron in a background with a temperature $T \ll m_e$ and zero chemical potential. Thus, the background contained only photons, but not electrons or any other matter particles. The dispersion relation for an electron with momentum $\vec{P}$ in the rest frame of the medium is given by

$$E_e(P) = \sqrt{P^2 + m_e^2 + \frac{2}{3}\alpha\pi T^2} \tag{1.1}$$

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and $M_e = E_e(0)$. Thus, to $O(\alpha)$,

$$M_e = m_e + \frac{\alpha\pi T^2}{3m_e} . \tag{1.2}$$

In the same reference frame, the gravitational mass was calculated to be

$$M'_e = M_e \left(1 - \frac{2\alpha\pi T^2}{3M_e^2}\right) . \tag{1.3}$$

Using Eq. (1.2) and keeping only terms up to $O(\alpha)$, this can be rewritten as

$$M'_e = m_e - \frac{\alpha\pi T^2}{3m_e} , \tag{1.4}$$

which is different from the inertial mass.

Moreover, although in those calculations only the case of the electron was considered explicitly, the above results are equally applicable to other charged fermions, such as the muon. In particular, we note that for any such fermion f, the ratio

$$\frac{M'_f}{M_f} = 1 - \frac{2\alpha\pi T^2}{3m_f^2} + O(\alpha^2) , \tag{1.5}$$

depends on the mass parameter m_f. Therefore, not only the inertial and gravitational masses of a given fermion cease to be equal when the background effects are taken into account, but in addition the ratio of these two quantities is no longer the same for all the particles; i.e., universality is lost as well. This happens despite the fact that the background contains only photons and is therefore flavor neutral. The origin of this difference is that while the background as well as the tree-level gravitational couplings are flavor-independent, the mass terms in the kinetic energy part of the Lagrangian are not. We should not be too surprised by this fact because, for example, even in the vacuum the anomalous magnetic moment has different contributions for the muon and the electron, though not at the lowest order.

But in a matter background with a non-zero chemical potential, such as the Sun or a supernova, there are contributions to gravitational mass which are proportional to the electron and nucleon densities. These matter contributions can dominate over the photon-background contribution, even when $T \ll m_e$, for which the photon contribution becomes negligible. Moreover, the matter-induced corrections to the gravitational mass will be different for the various charged lepton flavors, and will not be the same for the corresponding antiparticles.

Motivated by these considerations, in this work we calculate the leading matter-induced QED corrections to the gravitational masses of charged fermions in a medium that consists of a photon background and a matter background of electrons and nucleons. These represent the dominant corrections for charged leptons and antileptons. For strongly interacting particles such as the quarks, gluon exchange corrections are expected to be even stronger and our results will not apply.

Our calculation is based on the one-loop corrections to the gravitational vertex function of the charged lepton in the medium. Working in the context of the linearized theory of gravity, we show in detail how the gravitational mass is determined from the gravitational vertex function, give general expressions for the matter-induced corrections to the gravitational mass in a generic matter background, and give explicit formulas for the corrections in terms of the macroscopic parameters of the background medium for a few special cases of the background gases.

The rest of the paper is organized as follows. In Sec. 2, we discuss the general procedure for finding the inertial and gravitational masses. In Sec. 3, we discuss the self-energy diagrams for the charged leptons in a medium and find the medium-induced contributions to their inertial masses. We also calculate the wave function normalization factors which will be needed in the calculation of the gravitational mass later. In Sec. 4, we discuss the couplings in the linearized theory of gravity and calculate the gravitational vertex of the leptons. In Sec. 5, we use the vertex to find the gravitational masses of charged leptons and antileptons in a medium. The terms involving fermion distribution functions cannot be evaluated exactly. In Sec. 6, we evaluate the corrections in two different limits, viz., the classical and the strongly degenerate limit for the electron gas. Sec. 7 contains our concluding remarks.

2 Preliminaries

2.1 Inertial mass

The dispersion relations of the modes that propagate through the medium are determined by solving the linear part of the effective field equation. For fermions that propagate with momentum p^μ, this equation, in momentum space, is

$$\Big(\not{p} - m_f - \Sigma_f(p)\Big)\xi(p) = 0\,, \tag{2.1}$$

where Σ_f denotes the background-dependent part of the self-energy. The dispersion relations of the particle and the antiparticle are given by the positive and negative energy solutions of Eq. (2.1), and we denote the corresponding spinors by $U(p) = \xi(p)$ and $V(p) = \xi(-p)$, respectively.

In an isotropic medium, the most general form of Σ_f is

$$\Sigma_f(p) = a\not{p} + b\not{v} + c\,, \tag{2.2}$$

where we have introduced the vector v^μ which represents the velocity four-vector of the medium. We will perform all calculations in the rest frame of the medium, in which v^μ has components

$$v^\mu = (1, \vec{0}\,)\,, \tag{2.3}$$

and in that frame, we define the components of p^μ by writing

$$p^\mu = (p^0, \vec{P})\,. \tag{2.4}$$

In general, a, b, c are functions of the variables p^0 and P, which we will indicate by writing them as $a(p^0, P)$, and similarly for the other ones, when we need to show it explicitly. Eq. (2.2) can contain an additional term proportional to $\sigma^{\mu\nu}v_\mu p_\nu$ in the more general case. However, such a term does not appear at the level of the one-loop calculations [3] that we are considering in this work, and therefore we omit it.

Requiring Eq. (2.1) to have non-trivial solutions yields the condition

$$D(p^0, \vec{P}) = 0 \tag{2.5}$$

where

$$D(p^0, \vec{P}) = [(1-a)p - bv]^2 - (m_f + c)^2\,. \tag{2.6}$$

Eq. (2.5) also determines the poles of the fermion propagator

$$S'_f(p) = \frac{1}{p - m_f - \Sigma_f}\,, \tag{2.7}$$

which can be written in the form

$$S'_f = \frac{N(p_0, \vec{P})}{D(p_0, \vec{P})}\,, \tag{2.8}$$

with

$$N(p^0, \vec{P}) = (1-a)\not{p} - b\not{v} + (m_f + c)\,. \tag{2.9}$$

The condition given in Eq. (2.5) has a positive energy solution corresponding to the particle, given by $p^0 = E_f(P)$, and a negative energy solution corresponding to the antiparticle given by $p^0 = -E_{\bar{f}}(P)$, where

$$E_{f,\bar{f}}(P) = \sqrt{P^2 + \left(\frac{m_f + c}{1-a}\right)^2} \pm \frac{b}{1-a}\,. \tag{2.10}$$

These are implicit equations for $E_{f,\bar{f}}$ as a function of P. While solving for E_f, for example, we need to take the quantities a, b, c appearing on the right side as functions of E_f and P. The corresponding inertial masses are then defined as

$$M_{f,\bar{f}} = E_{f,\bar{f}}(0)\,. \tag{2.11}$$

Since a, b, c are of $O(e^2)$, we can solve Eq. (2.10) perturbatively by substituting the tree-level value $p^0 = \pm\sqrt{P^2 + m_f^2}$ in the right-hand side. It is useful to introduce the notation

$$\mathcal{E}_{f,\bar{f}}(p^0, \vec{P}) \equiv (ap\cdot v + b) \pm c\,, \tag{2.12}$$

which can be expressed concisely in terms of Σ_f as

$$\mathcal{E}_{f,\bar{f}} = \frac{1}{4}\mathrm{Tr}\left[(\not{v} \pm 1)\Sigma_f\right]. \tag{2.13}$$

To $O(e^2)$, the inertial masses are then found to be given by

$$\begin{aligned} M_f &= m_f + \mathcal{E}_f(m_f, \vec{0})\,, \\ M_{\bar{f}} &= m_f - \mathcal{E}_{\bar{f}}(-m_f, \vec{0})\,. \end{aligned} \tag{2.14}$$

Equation (2.13) is a useful formula that allows us to extract the matter-induced corrections to the inertial mass directly from the one-loop expression for Σ_f. As we will see next, the wavefunction renormalization factor is determined in terms of the same quantities $\mathcal{E}_f$ and $\mathcal{E}_{\bar{f}}$.

2.2 Wave function

We consider in some detail the case of the particles, and summarize at the end the corresponding results for the antiparticles. We adopt the normalization of the one-particle states such that their state vectors $|f(p,s)\rangle$ satisfy

$$\langle f(p',s')|f(p,s)\rangle = (2\pi)^3\delta^{(3)}(\vec{P} - \vec{P}')\delta_{s,s'}\,. \tag{2.15}$$

The one-particle states have associated with them the wave functions defined by the matrix element of the field operator

$$\left\langle 0\middle|\psi(x)\middle|f(p,s)\right\rangle = \sqrt{Z_f(p)}\,U_s(p)e^{-ip\cdot x}\,, \tag{2.16}$$

where $U_s(p)$ satisfies the Dirac equation

$$\left(\not{p} - m_f - \Sigma_f(p)\right)U_s(p) = 0 \tag{2.17}$$

with $p^\mu = (E_f(P), \vec{P})$. In the rest frame of the medium, the explicit form of $U_s(p)$ can be easily worked out. Adopting that frame, and choosing the normalization such that

$$U_s^\dagger(p)U_s(p) = 1\,, \tag{2.18}$$

it then follows that the U_s satisfy the spinor sum relation

$$\sum_s U_s(p)\overline{U}_s(p) = \frac{N(E_f,\vec{P})}{2\left[(1-a)E_f - b\right]} \tag{2.19}$$

where N is defined in Eq. (2.9). From Eq. (2.17) we obtain the identity

$$\overline{U}_s(p)\gamma_\mu U_s(p) = \left[\frac{(1-a)p_\mu - bv_\mu}{m_f + c}\right]\,\overline{U}_s(p)U_s(p) \tag{2.20}$$

which, together with Eq. (2.18) imply the relations

$$\overline{U}_s(p)U_s(p) = \frac{m_f + c}{(1-a)E_f - b} \tag{2.21}$$

and

$$\overline{U}_s(p)\gamma_\mu U_s(p) = \frac{(1-a)p_\mu - bv_\mu}{(1-a)E_f - b}\,. \tag{2.22}$$

In particular, in the frame specified by Eq. (2.3),

$$\begin{aligned}\left[\overline{U}(p)U(p)\right]_{\vec{P}=0} &= 1\,,\\ \left[\overline{U}(p)\gamma_\mu U(p)\right]_{\vec{P}=0} &= v_\mu\,.\end{aligned} \tag{2.23}$$

The normalization factor Z_f that appears in Eq. (2.16) is determined as follows. Near the pole $p^0 = E_f(P)$, Eq. (2.8) reduces to

$$S'_f(p) \approx \frac{N(E_f,\vec{P})}{(p_0 - E_f)\left(\frac{\partial D}{\partial p_0}\right)_{p^0=E_f}}\,. \tag{2.24}$$

On the other hand, we can calculate the one-particle contribution to the thermal propagator $iS'_f(x) = \langle T\psi(x)\overline{\psi}(0)\rangle$ by inserting a complete set of states, and retaining only the matrix elements between the vacuum state and one-particle states. Using Eq. (2.16), we obtain

$$S'_f(p)\Big|_{1-\text{particle}} \approx \frac{Z_f(p)\sum_s U_s(p)\overline{U}_s(p)}{p^0 - E_f} \tag{2.25}$$

near the same pole. The requirement that the residues of these two expressions coincide, then yields

$$Z_f(p) = \left\{ 2\Big[(1-a)E_f - b\Big] \left(\frac{\partial D}{\partial p_0}\right)^{-1} \right\}_{p^0 = E_f}, \tag{2.26}$$

where we have used Eq. (2.19). To the lowest order in e^2, and for the particular case $\vec{P} = 0$ in which we are interested, the expression reduces to

$$Z_f = 1 + \zeta_f \,, \tag{2.27}$$

where

$$\zeta_f = \left.\frac{\partial \mathcal{E}_f}{\partial p^0}\right|_{p^\mu = (m_f, \vec{0})}, \tag{2.28}$$

with $\mathcal{E}_f$ given by Eq. (2.12) or (2.13). From now on whenever we omit the dependence of Z_f on p, it is to be understood as the quantity evaluated at $\vec{P} = 0$.

For the case of the antiparticle, similar considerations apply. The wavefunction for the antiparticles is defined by

$$\Big\langle \bar{f}(p,s) \Big| \psi(x) \Big| 0 \Big\rangle = \sqrt{Z_{\bar{f}}(p)}\; V_s(p) e^{ip\cdot x} \,, \tag{2.29}$$

where $V_s(p)$ satisfies the equation

$$\Big(\not{p} + m_f + \Sigma_f(-p)\Big) V_s(p) = 0 \,, \tag{2.30}$$

with the normalization

$$V_s^\dagger(p) V_s(p) = 1 \,, \tag{2.31}$$

and $p^\mu = (E_{\bar{f}}(P), \vec{P})$. The analogy of Eq. (2.20) in the present case is

$$\overline{V}_s(p)\gamma_\mu V_s(p) = -\left[\frac{(1 - a(-p))p_\mu + b(-p)v_\mu}{m + c(-p)}\right] \overline{V}_s(p) V_s(p) \,. \tag{2.32}$$

Writing

$$Z_{\bar{f}} = 1 + \zeta_{\bar{f}} \,, \tag{2.33}$$

the same procedure that lead to Eq. (2.28) leads to the formula

$$\zeta_{\bar{f}} = \left.\frac{\partial \mathcal{E}_{\bar{f}}}{\partial p^0}\right|_{p^\mu = (-m_f, \vec{0})}. \tag{2.34}$$

Eqs. (2.28) and (2.34) are the formulas that we will use for the explicit calculations in Sec. 4.

We will denote by u_s and v_s the limiting value of the spinors U_s and V_s when the effects of the medium are neglected. They satisfy the free Dirac equation in the vacuum, as well as the relations

$$\bar{u}_s\gamma_\mu u_s = \frac{p_\mu}{m_f}\bar{u}_s u_s \tag{2.35}$$

$$\bar{u}_s u_s = \frac{m_f}{E_f} \,, \tag{2.36}$$

with similar relations for v_s but with the substitution $p_\mu \to -p_\mu$ in the above equations.

2.3 Gravitational mass

The gravitational mass is a measure of the strength of the coupling of the fermion to the graviton. It can be determined in terms of the fermion's vertex function for the gravitational interaction, as follows.

We denote by $\Gamma_{\lambda\rho}(p,p')$ the one-particle irreducible vertex function, defined such that the matrix element of the total stress-energy tensor operator $\widehat{T}_{\lambda\rho}(x)$ between incoming and outgoing fermion states is given by

$$\left\langle f(p',s')\middle|\widehat{T}_{\lambda\rho}(0)\middle|f(p,s)\right\rangle = \sqrt{Z_f(p)Z_f(p')}\,\overline{U}_{s'}(p')\Gamma_{\lambda\rho}(p,p')U_s(p)\,. \tag{2.37}$$

We perform all our calculations in the linearized theory of gravity. This means that we write

$$g_{\lambda\rho} = \eta_{\lambda\rho} + 2\kappa h_{\lambda\rho}\,, \tag{2.38}$$

and then $h_{\lambda\rho}$ is identified with the graviton field and treated as a weak field. κ is related to Newton's constant G through the equation

$$\kappa = \sqrt{8\pi G} \tag{2.39}$$

to ensure that the graviton field has the correctly normalized kinetic energy term in the Lagrangian. We write the complete vertex function in the form

$$\Gamma_{\lambda\rho} = V_{\lambda\rho} + \Gamma'_{\lambda\rho}\,, \tag{2.40}$$

where $\Gamma'_{\lambda\rho}$ denotes the 1-loop contribution and $V_{\lambda\rho}$ is the tree-level vertex function given by [4, 5]

$$V_{\lambda\rho}(p,p') = \frac{1}{4}\left[\gamma_\lambda(p+p')_\rho + \gamma_\rho(p+p')_\lambda\right] - \frac{1}{2}\eta_{\lambda\rho}\left[(\not{p} - m_f) + (\not{p}' - m_f)\right]\,. \tag{2.41}$$

We now consider the scattering of the fermion off a static gravitational potential, which is produced by a static mass density $\rho^{\rm ext}(\vec{x})$. Defining the Fourier transform

$$\phi^{\rm ext}(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\phi^{\rm ext}(\vec{q})e^{i\vec{q}\cdot\vec{x}}\,, \tag{2.42}$$

with a similar definition for $\rho^{\rm ext}(\vec{q})$, the corresponding metric is such that, in momentum space,

$$h^{\lambda\rho}(\vec{q}) = \frac{1}{\kappa}\phi^{\rm ext}(\vec{q})\left(2v^\lambda v^\rho - \eta^{\lambda\rho}\right)\,, \tag{2.43}$$

where we have used the Poisson equation $-2\vec{q}^{\,2}\phi^{\rm ext} = \kappa^2\rho^{\rm ext}$. The formula in Eq. (2.43) is the solution to the linearized field equation for the metric with the static energy momentum tensor $T^{\lambda\rho} = v^\lambda v^\rho \rho^{\rm ext}$, where $\rho^{\rm ext}$ is independent of time. Under the influence of such an external potential, the on-shell $f\to f$ transition amplitude is then

$$S_{ff} = -i\kappa(2\pi)\delta(E_f - E'_f)\sqrt{Z_f(p)Z_f(p')}\left[\overline{U}_s(p')\Gamma_{\lambda\rho}(p,p')U_s(p)\right]h^{\lambda\rho}(\vec{P}-\vec{P}')\,, \tag{2.44}$$

Substituting Eq. (2.43) in (2.44) yields

$$S_{ff} = -i(2\pi)\delta(E_f - E'_f)\mathcal{M}(\vec{P},\vec{P}')\phi^{\rm ext}(\vec{P}-\vec{P}')\,, \tag{2.45}$$

where we have defined

$$\mathcal{M}(\vec{P},\vec{P}') \equiv (2v^\lambda v^\rho - \eta^{\lambda\rho})\sqrt{Z_f(p)Z_f(p')}\Big[\overline{U}_s(p')\Gamma_{\lambda\rho}(p,p')U_s(p)\Big]_{E'_f=E_f} . \tag{2.46}$$

$\mathcal{M}(\vec{P},\vec{P}')$ is essentially the off-diagonal element of the Fourier transform of the mass operator, and the gravitational mass is simply the value of this quantity when both initial and final fermions have vanishing 3-momentum,

$$M'_f \equiv \lim_{\vec{P}\to 0}\Big[\mathcal{M}(\vec{P},\vec{P}')\Big]_{\vec{P}'\to\vec{P}} . \tag{2.47}$$

To justify more fully this identification, notice that the mass density operator for the fermion, $\rho_f(t,\vec{x})$, is determined by writing an effective Lagrangian

$$\mathscr{L}_{\rm eff} = -\rho_f(t,\vec{x})\phi^{\rm ext}(\vec{x}) \tag{2.48}$$

such that Eq. (2.45) is reproduced by taking the S-matrix element using $\mathscr{L}_{\rm eff}$ as the interaction Lagrangian. This gives the scattering amplitude

$$\left\langle f(p',s)\left|\int d^4x\,(i\mathscr{L}_{\rm eff})\right| f(p,s)\right\rangle = -i2\pi\delta(E'_f - E_f)\left\langle f(p',s)\left|\rho(0,\vec{0})\right| f(p,s)\right\rangle \phi^{\rm ext}(\vec{P}-\vec{P}') . \tag{2.49}$$

Comparison with Eq. (2.45) shows that ρ_f is such that

$$\left\langle f(p',s)\left|\rho_f(0,\vec{0})\right| f(p,s)\right\rangle_{P'=P} = \mathcal{M}(\vec{P},\vec{P}') \tag{2.50}$$

with $\mathcal{M}(\vec{P},\vec{P}')$ given in Eq. (2.46). By definition, the gravitational mass M'_f is given by

$$\left[\left\langle f(p',s)\left|\int d^3x\,\rho(0,\vec{x})\right| f(p,s)\right\rangle\right]_{\vec{P}\to 0} = (2\pi)^3\delta^{(3)}(\vec{P}-\vec{P}\,')M'_f \tag{2.51}$$

while, on the other hand,

$$\left\langle f(p',s)\left|\int d^3x\rho(0,\vec{x})\right| f(p,s)\right\rangle = (2\pi)^3\delta^{(3)}(\vec{P}-\vec{P}')\left\langle f(p',s)\left|\rho(0,\vec{0})\right| f(p,s)\right\rangle . \tag{2.52}$$

Comparing Eqs. (2.51) and (2.52), and using (2.50), we arrive at the formula given in Eq. (2.47).

2.4 Operational definition at $O(e^2)$

Using Eqs. (2.40) and (2.27), the formula given by Eqs. (2.46) and (2.47) can be rewritten in the form

$$\begin{aligned} M'_f &= (2v^\lambda v^\rho - \eta^{\lambda\rho}) \\ &\times \lim_{\vec{P}\to 0}\left\{\Big[\overline{U}_s(p')\Big\{V_{\lambda\rho}(p,p') + \zeta_f V_{\lambda\rho}(p,p') + Z_f\Gamma'_{\lambda\rho}(p,p')\Big\}U_s(p)\Big]_{\substack{E'_f=E_f\\ \vec{P}\,'\to\vec{P}}}\right\} . \end{aligned} \tag{2.53}$$

Since ζ_f and $\Gamma'_{\lambda\rho}$ are $O(e^2)$, in any term that contains either of these factors we substitute the tree level expressions for the other quantities. Furthermore, the terms involving $V_{\lambda\rho}$ can be evaluated

immediately with the help of the identities given in Eq. (2.23). Remembering that $E_f(0) = M_f$, we finally obtain the operational definition to $O(e^2)$

$$M'_f = 3M_f - 2m_f + \zeta_f m_f + (2v^\lambda v^\rho - \eta^{\lambda\rho}) \lim_{\vec{P}\to 0} \left\{ \left[\overline{u}_s(p')\Gamma'_{\lambda\rho}(p,p')u_s(p) \right]_{\substack{E'_f = E_f \\ \vec{P}' \to \vec{P}}} \right\}, \tag{2.54}$$

where we can set $E_f = \sqrt{P^2 + m_f^2}$ in the last term.

The arguments for the case of the antiparticle are similar, but the equation corresponding to Eq. (2.44) is

$$S_{\bar{f}\bar{f}} = (-1)(-i\kappa)(2\pi)\delta(E_{\bar{f}} - E'_{\bar{f}})\sqrt{Z_{\bar{f}}(p)Z_{\bar{f}}(p')}\, \left[\overline{V}_s(p)\Gamma_{\lambda\rho}(-p',-p)V_s(p') \right] h^{\lambda\rho}(\vec{P} - \vec{P}'), \tag{2.55}$$

where the extra minus sign is due to the usual fermion exchange rule. This leads to an equation that is analogous to Eq. (2.53), but with an extra minus sign in front and some obvious changes in the corresponding symbols, which in turn lead to the $O(e^2)$ formula

$$M'_{\bar{f}} \;=\; 3M_{\bar{f}} - 2m_f + \zeta_{\bar{f}} m_f - (2v^\lambda v^\rho - \eta^{\lambda\rho}) \lim_{\vec{P}\to 0} \left\{ \left[\overline{v}_s(p)\Gamma'_{\lambda\rho}(-p',-p)v_s(p') \right]_{\substack{E'_f = E_f \\ \vec{P}' \to \vec{P}}} \right\}. \tag{2.56}$$

For the following discussion, it is useful to indicate explicitly the dependence of the vertex function $\Gamma'_{\lambda\rho}(p,p')$ on the vector v^μ, and therefore we will write as $\Gamma'_{\lambda\rho}(p,p',v)$. Using the usual relation between the free particle and antiparticle spinors by means of the charge conjugation matrix C, the spinor matrix element that appears in Eq. (2.56) can be rewritten in the form

$$\overline{v}_s(p)\Gamma'_{\lambda\rho}(-p',-p,v)v_s(p') = -\overline{u}_s(p')\Gamma'^{c}_{\lambda\rho}(-p',-p,v)u_s(p)\,, \tag{2.57}$$

where, for any 4×4 matrix A, we define

$$A^c \equiv C^{-1}A^T C\,. \tag{2.58}$$

On the other hand, the following result holds. If the Lagrangian of the theory is C invariant (which in our case it is) and if the background is C-symmetric, then the gravitational vertex function satisfies the relation

$$\Gamma'^{c}_{\lambda\rho}(-p',-p,v) = \Gamma'_{\lambda\rho}(p,p',v)\,. \tag{2.59}$$

This result is obtained by the same techniques that were employed in Ref. [6] to analyze the transformation properties of the induced electromagnetic vertex of neutrinos in a matter background. This result cannot be applied in our case because we will consider backgrounds which are not particle-antiparticle asymmetric. However, as an extension of Eq. (2.59), similar arguments can be used to show that, if the Lagrangian is C invariant but the background is not C-symmetric, then the vertex function satisfies

$$\Gamma'^{c}_{\lambda\rho}(-p',-p,v) = \Gamma'_{\lambda\rho}(p,p',-v)\,. \tag{2.60}$$

Using Eq. (2.60) in (2.57) and substituting the result in Eq. (2.56), we then obtain the formula

$$M'_{\bar{f}} = 3M_{\bar{f}} - 2m_f + \zeta_{\bar{f}} m_f + (2v^\lambda v^\rho - \eta^{\lambda\rho}) \lim_{\vec{P}\to 0} \left\{ \left[\overline{u}_s(p')\Gamma'_{\lambda\rho}(p,p',-v)u_s(p) \right]_{\substack{E'_f = E_f \\ \vec{P}' \to \vec{P}}} \right\}. \tag{2.61}$$

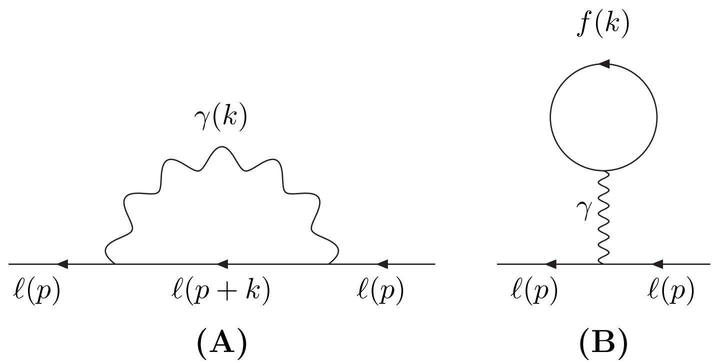

Figure 1: One-loop diagrams for the self-energy of a charged lepton ℓ in a medium.

We take the opportunity to emphasize the following point. In the calculations that follow, we will find expressions for the various contributions to $\Gamma'_{\lambda\rho}(p,p')$, which are given as integrals over the propagators and thermal distribution functions. In general, such expressions do not have a unique limiting value as we let $p' \to p$ in an arbitrary way [7]. Moreover, some of the integrals are ill-defined if the limit is not taken properly. In our case, the precise order in which the various limits must be taken has been dictated by the physical issue at hand. Thus, since we are interested in the interaction of the particle with a static gravitational potential, the quantity that enters is $\Gamma'_{\lambda\rho}(p,p')$, evaluated for $E'_f = E_f$. Next we set $\vec{P}' = \vec{P}$ since we actually want the forward scattering amplitude, and finally set $\vec{P} \to 0$ to obtain the coupling at zero momentum, which determines the gravitational mass. This justifies the somewhat cumbersome notation regarding the limits in Eq. (2.54), but it is meant to indicate precisely what we have just explained, since failure to follow this prescription results in ill-defined expressions in some contributions. On the other hand, as we will see, this prescription allowed us to evaluate all the integrals involved, in a unique and well-defined way, including those that superficially seem to be singular, without having to introduce by hand any special regularization technique.

3 Self-energy

3.1 Calculation of $\mathcal{E}_\ell$

The self-energy diagrams are shown in Fig. 1. In the absence of a gravitational potential, the contribution from Fig. 1B vanishes because the photon tadpole is zero in an electrically neutral medium [8]. In the presence of a gravitational potential, that diagram is not zero by itself because the condition for the vanishing of the photon tadpole, which is equivalent to require that the medium be electrically neutral, involves other diagrams. This will be discussed in detail in Sec. 4.3. As shown there, the conclusion remains that we need to consider only Fig. 1A to calculate the self-energy.

Therefore, the charged lepton self-energy is given by

$$-i\Sigma_\ell(p) = (ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu i S_\ell(p+k)\gamma^\nu i D_{\mu\nu}(k)\,, \tag{3.1}$$

where $S_\ell(k)$ and $D_{\mu\nu}(k)$ are the thermal propagators for the internal lines. For a fermion, the propagator is given by

$$iS_f(p) = iS_{Ff}(p) + S_{Tf}(p) \tag{3.2}$$

where

$$
\begin{aligned}
S_{Ff} &= \frac{\not{p} + m_f}{p^2 - m_f^2 + i\epsilon}\,, &(3.3)\\
S_{Tf}(p) &= -2\pi(\not{p} + m_f)\delta(p^2 - m_f^2)\eta_f(p)\,. &(3.4)
\end{aligned}
$$

For the photon, in the Feynman gauge,

$$
iD_{\lambda\rho}(k) = -\eta_{\lambda\rho}\left[i\Delta_F(k) + \Delta_T(k)\right]\,, \tag{3.5}
$$

where

$$
\begin{aligned}
\Delta_F(k) &= \frac{1}{k^2 + i\epsilon}\,, &(3.6)\\
\Delta_T(k) &= 2\pi\delta(k^2)\eta_\gamma(k)\,. &(3.7)
\end{aligned}
$$

We have introduced the notation

$$
\begin{aligned}
\eta_f(p) &= \frac{\theta(p\cdot v)}{e^{\beta(p\cdot v - \mu_f)} + 1} + \frac{\theta(-p\cdot v)}{e^{-\beta(p\cdot v - \mu_f)} + 1}\,, &(3.8)\\
\eta_\gamma(k) &= \frac{1}{e^{\beta\,|k\cdot v|} - 1}\,. &(3.9)
\end{aligned}
$$

where $\beta = 1/T$ is the inverse temperature of the background and μ_f the chemical potential.

When Eqs. (3.2) and (3.5) are substituted into Eq. (3.1), four terms are produced. Since we are interested in the background induced contributions only, we disregard the term involving both $S_{F\ell}$ and Δ_F. Among the other three, the one involving both $S_{T\ell}$ and Δ_T contributes only to the absorptive part of the self-energy — i.e., to the imaginary part of the coefficients a, b, c in Eq. (2.2) — and therefore do not contribute to the mass. The contributions to the real part of the coefficients arises from the remaining two terms, which can be written in the form

$$
\Sigma'_\ell(p) = \Sigma'_{\ell 1}(p) + \Sigma'_{\ell 2}(p)\,, \tag{3.10}
$$

where

$$
\begin{aligned}
\Sigma'_{\ell 1}(p) &= 2e^2 \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\frac{\not{p} + \not{k} - 2m_\ell}{p^2 + 2k\cdot p - m_\ell^2}\\
\Sigma'_{\ell 2}(p) &= -\,2e^2 \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m_\ell^2)\eta_\ell(k)\,\frac{\not{k} - 2m_\ell}{p^2 - 2k\cdot p + m_\ell^2}\,. &(3.11)
\end{aligned}
$$

Using Eq. (2.13), and according to the decomposition given in Eq. (3.10), we write

$$
\mathcal{E}_\ell = \mathcal{E}_{\ell 1} + \mathcal{E}_{\ell 2}\,, \tag{3.12}
$$

where

$$
\begin{aligned}
\mathcal{E}_{\ell 1} &= 2e^2 \int \frac{d^4k}{(2\pi)^3}\delta(k^2)\eta_\gamma(k)\frac{p\cdot v + k\cdot v - 2m_\ell}{p^2 + 2k\cdot p - m_\ell^2}\,, &(3.13)\\
\mathcal{E}_{\ell 2} &= -2e^2 \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m_\ell^2)\eta_\ell(k)\frac{k\cdot v - 2m_\ell}{p^2 - 2k\cdot p + m_\ell^2}\,. &(3.14)
\end{aligned}
$$

We can make a similar decomposition of $\mathcal{E}_{\bar{\ell}}$. The quantities $\mathcal{E}_{\bar{\ell}1}$ and $\mathcal{E}_{\bar{\ell}2}$ are obtained from $\mathcal{E}_{\ell 1}$ and $\mathcal{E}_{\ell 2}$ by replacing m_ℓ by $-m_\ell$.

3.2 Inertial mass

The inertial mass is determined by applying Eq. (2.14) and, according to the decomposition given in Eq. (3.12) we write it as

$$M_\ell = m_\ell + m_{\ell 1} + m_{\ell 2} \,, \tag{3.15}$$

and similarly for the anti-leptons. Substituting $p^\mu = (m_\ell, \vec{0})$ in Eq. (3.13), and using the fact that the terms in the integrand that are odd in k yield zero, we obtain

$$\begin{aligned} m_{\ell 1} \equiv \mathcal{E}_{\ell 1}(m_\ell, \vec{0}) &= \frac{e^2}{m_\ell} \int \frac{d^4k}{(2\pi)^3} \delta(k^2) \eta_\gamma(k) \\ &= \frac{e^2 T^2}{12 m_\ell} \,. \end{aligned} \tag{3.16}$$

This is the contribution to the inertial mass from the photons in the background, in agreement with the result quoted in Eq. (1.2), and it is non-zero for any the charged lepton propagating through the medium. In a similar fashion we find

$$m_{\bar{\ell}1} \equiv -\mathcal{E}_{\bar{\ell}1}(-m_\ell, \vec{0}) = \frac{e^2 T^2}{12 m_\ell} \,, \tag{3.17}$$

and therefore the photon contribution for the anti-particle is the same as for the corresponding particles.

The term given in Eq. (3.14) is due to the fermions in the background. Therefore in a background that contains electrons but not the other charged leptons, the distribution functions for the muon and the tau vanish. As a result,

$$m_{\mu 2} = m_{\tau 2} = m_{\bar{\mu}2} = m_{\bar{\tau}2} = 0 \,. \tag{3.18}$$

For the electron, we obtain

$$m_{e2} \equiv \mathcal{E}_{e2}(m_e, \vec{0}) = \frac{e^2}{m_e} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m_e^2) \eta_e(k) \left[\frac{k_0 - 2m_e}{k_0 - m_e} \right] \tag{3.19}$$

Performing the integrations over k_0 and the angular variables, we obtain

$$m_{e2} = \frac{e^2}{2\pi^2 m_e} \int_0^\infty dK \frac{K^2}{2E_K} \left[\left(\frac{E_K - 2m_e}{E_K - m_e} \right) f_e(E_K) + \left(\frac{E_K + 2m_e}{E_K + m_e} \right) f_{\bar{e}}(E_K) \right] \,, \tag{3.20}$$

where we have put

$$k^\mu = (E_K, \vec{K}) \,, \qquad E_K \equiv \sqrt{K^2 + m_e^2} \,, \tag{3.21}$$

and the distribution functions for a fermion and antifermion are given by the usual formulas

$$f_{f,\bar{f}}(E) = \frac{1}{e^{\beta(E \mp \mu_f)} + 1} \,, \tag{3.22}$$

respectively. Similarly,

$$\begin{aligned} m_{\bar{e}2} &\equiv -\mathcal{E}_{\bar{e}2}(-m_e, \vec{0}) \\ &= \frac{e^2}{2\pi^2 m_e} \int_0^\infty dK \frac{K^2}{2E_K} \left[\left(\frac{E_K + 2m_e}{E_K + m_e} \right) f_e(E_K) + \left(\frac{E_K - 2m_e}{E_K - m_e} \right) f_{\bar{e}}(E_K) \right] \,. \end{aligned} \tag{3.23}$$

The integration over K can be performed only when the momentum distribution functions are specified, and we will consider some examples in Sec. 6. Here we only note that, as it is expected on the basis of CPT-symmetry considerations, the inertial mass correction is the same for particle and anti-particle if the medium has zero chemical potential, but not otherwise.

3.3 Calculation of Z_ℓ

We decompose

$$\zeta_\ell = \zeta_{\ell 1} + \zeta_{\ell 2} \tag{3.24}$$

with a similar decomposition for the anti-leptons, where

$$\zeta_{\ell i} = \left.\frac{\partial \mathcal{E}_{\ell i}}{\partial p^0}\right|_{p^\mu=(m_\ell,\vec{0})} \qquad \text{for } i = 1, 2, \tag{3.25}$$

and

$$\zeta_{\bar{\ell} i} = \left.\frac{\partial \mathcal{E}_{\bar{\ell} i}}{\partial p^0}\right|_{p^\mu=(-m_\ell,\vec{0})} \qquad \text{for } i = 1, 2. \tag{3.26}$$

Taking the derivative in Eq. (3.13) and then setting $p^\mu = (\pm m_\ell, \vec{0})$, we obtain

$$\left.\frac{\partial \mathcal{E}_{\ell 1}}{\partial p^0}\right|_{p^\mu=(m_\ell,\vec{0})} = \left.\frac{\partial \mathcal{E}_{\bar{\ell} 1}}{\partial p^0}\right|_{p^\mu=(-m_\ell,\vec{0})} = -\frac{e^2}{m_\ell^2}\int \frac{d^3K}{(2\pi)^3}\frac{f_\gamma(K)}{K}\left(1 - \frac{m_\ell^2}{K^2}\right), \tag{3.27}$$

which implies

$$\zeta_{\ell 1} = \zeta_{\bar{\ell} 1} = -\frac{e^2 T^2}{12 m_\ell^2} + \frac{e^2}{2\pi^2}\int_0^\infty \frac{dK}{K}\, f_\gamma(K)\,, \tag{3.28}$$

where we have introduced the photon momentum distribution function

$$f_\gamma(K) = \frac{1}{e^{\beta K} - 1}\,. \tag{3.29}$$

The integral in Eq. (3.28) is infrared divergent, and it will cancel a similarly divergent term in the gravitational vertex contribution to the gravitational mass [See Eq. (5.5)].

Since the electron background terms do not contribute to the self-energy of the muon or the tau, it follows that

$$\zeta_{\mu 2} = \zeta_{\tau 2} = \zeta_{\bar{\mu} 2} = \zeta_{\bar{\tau} 2} = 0\,. \tag{3.30}$$

For the electron, Eq. (3.14) implies

$$\begin{aligned} \left.\frac{\partial \mathcal{E}_{e 2}}{\partial p^0}\right|_{p^\mu=(m_e,\vec{0})} &= -\frac{\mathcal{E}_e(m_e,\vec{0})}{m_e} \\ \left.\frac{\partial \mathcal{E}_{\bar{e} 2}}{\partial p^0}\right|_{p^\mu=(-m_e,\vec{0})} &= \frac{\mathcal{E}_{\bar{e}}(-m_e,\vec{0})}{m_e}\,, \end{aligned} \tag{3.31}$$

which yield

$$\begin{aligned} \zeta_{e 2} &= -\frac{m_{e 2}}{m_e} \\ \zeta_{\bar{e} 2} &= -\frac{m_{\bar{e} 2}}{m_e}\,, \end{aligned} \tag{3.32}$$

with m_{e2} and $m_{\bar{e}2}$ given in Eqs. (3.20) and (3.23), respectively.

4 Gravitational vertex

4.1 Irreducible diagrams and couplings

The irreducible one-loop diagrams for the vertex function are given in Figs. 2 and 3. We adopt the convention that q is the momentum of the outgoing graviton, so that

$$q = p - p' \,, \tag{4.1}$$

and we calculate only the terms that contribute to the the dispersive part of the vertex function, which satisfies the condition

$$\Gamma_{\lambda\rho}(p, p') = \gamma_0 \Gamma^{\dagger}_{\lambda\rho}(p', p)\gamma_0 \,. \tag{4.2}$$

The absorptive part contributes to the fermion damping, with which we are not concerned in the present work.

When the formulas given in Eqs. (3.2) and (3.5) for the propagators are substituted in the expressions corresponding to the diagrams, we obtain terms of different kind. One of them is independent of the background medium, in which we are not interested. Those involving two factors of the thermal part of the propagators contribute to the absorptive part of the vertex, while those involving three factors of the thermal part vanish because of the various δ-functions appearing in it. Thus, the background induced contribution to the dispersive part of the vertex, to be denoted by $\Gamma'_{\lambda\rho}$, contains the thermal part of only one of the propagators, and they are the only kind of term that we retain.

We have omitted the one-particle reducible diagrams in which the graviton line comes out from one of the external fermion legs, because they do not contribute to $\Gamma_{\lambda\rho}$. The proper way to take them into account in the calculation of the amplitude for any given process, is by choosing the external spinor to be the solution of the effective Dirac equation for the propagating fermion mode in the medium, instead of the spinor representing the free-particle solution of the equation in the vacuum, with the normalization determined by the self-energy of the fermion, as discussed in Sec. 2.2.

The various graviton couplings that are needed for the evaluation of these diagrams have been given earlier. For completeness we summarize here the relevant formulas. For fermions, the Feynman rule for the graviton-fermion-fermion vertex is $-i\kappa V_{\lambda\rho}(p, p')$, where $V_{\lambda\rho}$ given in Eq. (2.41), where p and p' are the momenta of the incoming and the outgoing fermions [4]. The interaction involving the graviton, a photon A^{μ} and a pair of charged fermions is represented by the Feynman rule [9, 5] $ie\kappa a_{\mu\nu\lambda\rho}\gamma^{\nu}$, where

$$a_{\mu\nu\lambda\rho} = \eta_{\mu\nu}\eta_{\lambda\rho} - \frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\nu\lambda}\eta_{\mu\rho}\right) \,. \tag{4.3}$$

In addition, there is also a photon-photon-graviton vertex. For an incoming photon $A_{\mu}(k)$ and an outgoing one $A^{\nu}(k')$, the Feynman rule for this vertex is $-i\kappa C_{\mu\nu\lambda\rho}(k, k')$, with [9]

$$\begin{aligned} C_{\mu\nu\lambda\rho}(k, k') \;=\; & \eta_{\lambda\rho}(\eta_{\mu\nu} k\cdot k' - k'_{\mu}k_{\nu}) - \eta_{\mu\nu}(k_{\lambda}k'_{\rho} + k'_{\lambda}k_{\rho}) \\ & + k_{\nu}(\eta_{\lambda\mu}k'_{\rho} + \eta_{\rho\mu}k'_{\lambda}) + k'_{\mu}(\eta_{\lambda\nu}k_{\rho} + \eta_{\rho\nu}k_{\lambda}) \\ & - k\cdot k'(\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu}) \,. \end{aligned} \tag{4.4}$$

4.2 Diagrams in Fig. 2

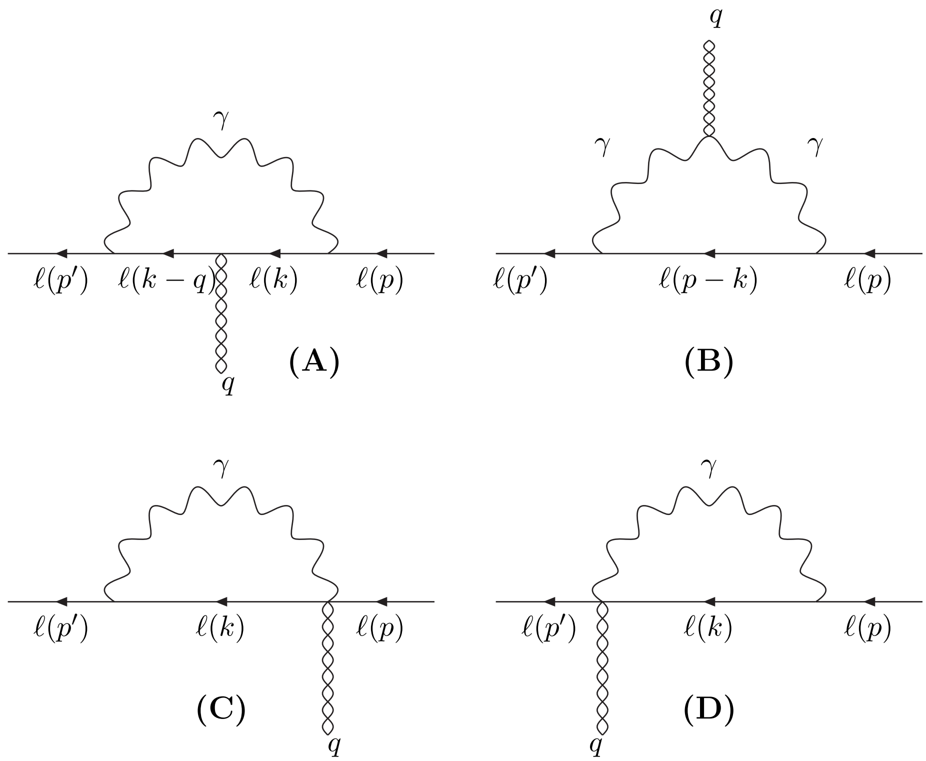

Figure 2: One-loop diagrams for the gravitational vertex of charged leptons in a background of electrons. The braided line represents the graviton.

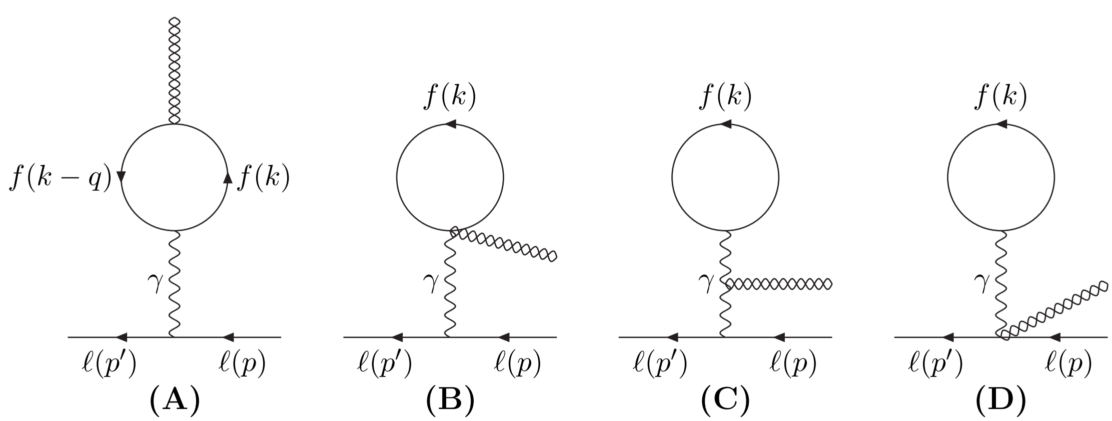

Figure 3: Diagrams for the one-loop contribution to the gravitational vertex of any charged lepton in a background of electrons and nucleons.

4.2.1 Diagram 2A

The amplitude of the diagram in Fig. 2A can be written as

$$-i\kappa\Gamma^{(A)}_{\lambda\rho}(p,p') = \int \frac{d^4k}{(2\pi)^4}\, ie\gamma_\alpha\, iS_\ell(k')\,(-i\kappa)V_{\lambda\rho}(k,k')\, iS_\ell(k)\, ie\gamma_\beta\, iD^{\alpha\beta}(k-p)\,, \tag{4.5}$$

where

$$k' \equiv k - q\,. \tag{4.6}$$

As already explained, to determine the contribution to the dispersive part of the vertex function we need to retain the terms that contain the thermal part of only one of the propagators. Any of them contains some combination of the form

$$\Lambda_{\lambda\rho}(k_1,k_2) \equiv \gamma_\alpha(\not{k}_2 + m_\ell)V_{\lambda\rho}(k_1,k_2)(\not{k}_1 + m_\ell)\gamma^\alpha\,. \tag{4.7}$$

After some straightforward algebra, this can be written as

$$\begin{aligned}\Lambda_{\lambda\rho}(k_1,k_2) &= -\frac{1}{2}\Big[(k_1+k_2)_\rho(\not{k}_1\gamma_\lambda\not{k}_2 + m_\ell^2\gamma_\lambda) + (k_1+k_2)_\lambda(\not{k}_1\gamma_\rho\not{k}_2 + m_\ell^2\gamma_\rho)\Big]\\ &\quad +\eta_{\lambda\rho}\Big[(k_1^2 - m_\ell^2)(\not{k}_2 - 2m_\ell) + (k_2^2 - m_\ell^2)(\not{k}_1 - 2m_\ell)\Big]\\ &\quad +2m_\ell(k_1+k_2)_\lambda(k_1+k_2)_\rho\,. \end{aligned}\tag{4.8}$$

For the sake of convenience, we divide the total contribution into two parts

$$\Gamma'^{(A)}_{\lambda\rho}(p,p') = \Gamma'^{(A1)}_{\lambda\rho}(p,p') + \Gamma'^{(A2)}_{\lambda\rho}(p,p')\,, \tag{4.9}$$

where $\Gamma'^{(A1)}_{\lambda\rho}$ contains the distribution function of the photon and therefore contributes to the gravitational vertex for all charged leptons, and $\Gamma'^{(A2)}_{\lambda\rho}$ contains the distribution function of the electrons and contributes only to the vertex for the electrons. Changing the integration variable from k to $k+p$, we obtain

$$\Gamma'^{(A1)}_{\lambda\rho}(p,p') = -e^2\int\frac{d^4k}{(2\pi)^3}\,\frac{\delta(k^2)\eta_\gamma(k)}{[(k+p')^2 - m_\ell^2][(k+p)^2 - m_\ell^2]}\Lambda_{\lambda\rho}(k+p,k+p')\,, \tag{4.10}$$

and similarly,

$$\begin{aligned}\Gamma'^{(A2)}_{\lambda\rho}(p,p') &= e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m_\ell^2)\eta_\ell(k)\\ &\quad\times\left(\frac{\Lambda_{\lambda\rho}(k,k-q)}{[(k-q)^2 - m_\ell^2](k-p)^2} + \frac{\Lambda_{\lambda\rho}(k+q,k)}{[(k+q)^2 - m_\ell^2](k-p')^2}\right).\end{aligned}\tag{4.11}$$

4.2.2 Diagram 2B

For this diagram

$$-i\kappa\Gamma^{(B)}_{\lambda\rho}(p,p') = \int\frac{d^4k}{(2\pi)^4}\, ie\gamma^\alpha\, iS_\ell(p-k)\, ie\gamma^\beta(-i\kappa)C_{\mu\nu\lambda\rho}(k,k')\, iD^{\nu\alpha}(k)iD^{\mu\beta}(k')\,, \tag{4.12}$$

and we decompose it in analogy with Eq. (4.9). The part that contains the photon distribution function is

$$\Gamma'^{(B1)}_{\lambda\rho}(p,p') = e^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\nu S_{F\ell}(p-k)\gamma^\mu C_{\mu\nu\lambda\rho}(k,k')\Big[\Delta_F(k)\Delta_T(k') + \Delta_F(k')\Delta_T(k)\Big]\,. \tag{4.13}$$

Making a change of the integration variable in one of the terms, this can be written as

$$\begin{aligned}\Gamma'^{(B1)}_{\lambda\rho}(p,p') &= e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\\ &\times\left[\frac{\gamma^\nu(\not p'-\not k+m_\ell)\gamma^\mu C_{\mu\nu\lambda\rho}(k+q,k)}{[(p'-k)^2-m_\ell^2](k+q)^2}+\frac{\gamma^\nu(\not p-\not k+m_\ell)\gamma^\mu C_{\mu\nu\lambda\rho}(k,k-q)}{[(p-k)^2-m_\ell^2](k-q)^2}\right]\end{aligned}\qquad(4.14)$$

while

$$\begin{aligned}\Gamma'^{(B2)}_{\lambda\rho}(p,p') &= e^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\nu S_{T\ell}(k)\gamma^\mu C_{\mu\nu\lambda\rho}(p-k,p'-k)\,\Delta_F(p-k)\Delta_F(p'-k)\\ &= -e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_\ell^2)\eta_\ell(k)\gamma^\nu(\not k+m_\ell)\gamma^\mu\,\frac{C_{\mu\nu\lambda\rho}(p-k,p'-k)}{(p-k)^2(p'-k)^2}\end{aligned}\qquad(4.15)$$

gives the lepton background part.

4.2.3 Diagrams 2C and 2D

For these two diagrams the manipulations are similar and, omitting the details, the results are

$$\Gamma'^{(C1+D1)}_{\lambda\rho}(p,p') = -\,e^2a_{\mu\nu\lambda\rho}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\left[\frac{\gamma^\mu(\not k+\not p'+m_\ell)\gamma^\nu}{(k+p')^2-m_\ell^2}+\frac{\gamma^\nu(\not k+\not p+m_\ell)\gamma^\mu}{(k+p)^2-m_\ell^2}\right]\qquad(4.16)$$

and

$$\Gamma'^{(C2+D2)}_{\lambda\rho}(p,p') = e^2a_{\mu\nu\lambda\rho}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_\ell^2)\eta_\ell(k)\left[\frac{\gamma^\mu(\not k+m_\ell)\gamma^\nu}{(k-p')^2}+\frac{\gamma^\nu(\not k+m_\ell)\gamma^\mu}{(k-p)^2}\right]\,.\qquad(4.17)$$

4.3 Diagrams in Fig. 3

4.3.1 The question of the photon tadpole

We are calculating the effective action given by the tree-level terms, plus the $O(e^2)$ corrections that arise from the diagrams in Figs. 1, 2 and 3. Some of the diagrams contribute to the bilinear (or kinetic) part of the effective action, from which we identify the inertial mass and the wavefunction renormalization, while others contribute to the interaction with the gravitational potential, from which we identify the gravitational mass.

It is important to recall at this point that we are considering a medium that is electrically neutral, which requires that the parameters that characterize the composition of the medium be such that the net contribution to the photon tadpole vanishes. The diagrams that contribute to the photon tadpole at the one-loop level, in the presence of a static and homogeneous gravitational potential, are shown in Fig. 4, where the graviton line represents represents the $q=0$ background field. In the absence of the background field, only the diagram 4A contributes to the photon tadpole. In that case, the requirement that the tadpole vanishes yields the familiar condition

$$\mathcal{Q}^{(4A)}\equiv\sum_f\mathcal{Q}_f\left[2\int\frac{d^3K}{(2\pi)^3}[f_f(E_K)-f_{\bar f}(E_K)]\right]=0\qquad(4.18)$$

where f_f and $f_{\bar f}$ are given by Eq. (3.22), and the sum is over all species of fermions in the medium, the charge of each species being denoted by $\mathcal{Q}_f$ with the convention that $\mathcal{Q}_e=-1$. In this case, the quantity $\mathcal{Q}^{(4A)}$ is identified with the total charge of the medium. However, in the presence

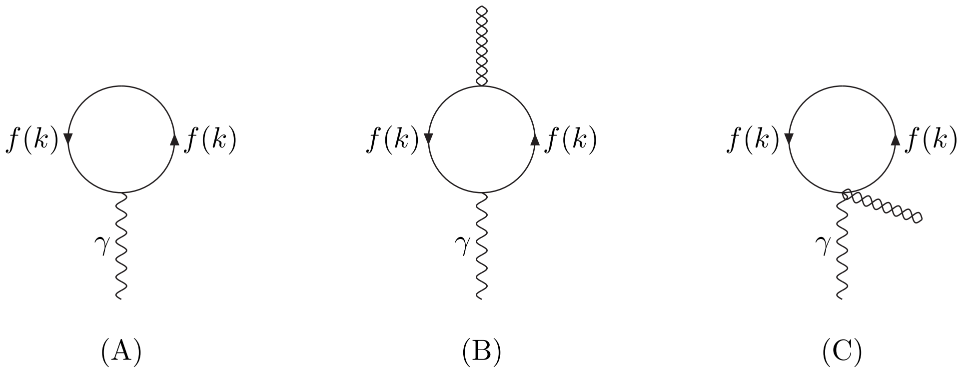

Figure 4: The one-loop diagrams that contribute to order κ to the photon tadpole in a medium, in the presence of a static and homogeneous ($q = 0$) gravitational potential. The fermion loop involves a sum over all the species of fermions present in the medium.

of the background field, and to the order that we are calculating, we have to take into account the contributions of the diagrams 4B and 4C to the photon tadpole or, equivalently, to the total charge of the system. If we denote them by $\mathcal{Q}^{(4B)}$ and $\mathcal{Q}^{(4C)}$ respectively, it is the sum $\mathcal{Q}^{(4A)} + \mathcal{Q}^{(4B)} + \mathcal{Q}^{(4C)}$ that must be zero for the photon tadpole to vanish. Physically, this means that the number density of the particles are not determined by their free distribution functions. The particle distributions rearrange themselves in a way that depends on the background gravitational field.

This has the following implication for our calculation. Firstly, the unadorned tadpole of Fig. 4A is now itself of order κ because of the charge neutrality condition. Since the diagrams 3C and 3D contain an explicit factor of κ apart from the unadorned tadpole, their contribution is actually of order κ^2 and therefore we can neglect them. Secondly, the diagram shown in Fig. 1B cancels the q-independent contributions from the diagrams 3A and 3B. Since the loop in diagram 3B in independent of q, this diagram is totally canceled.

In summary, the only contribution from the diagrams shown in Fig. 3 arises from the q^2-dependent part of the tadpole subdiagram of Fig. 3A. When multiplied by the photon propagator, it gives zero for the $\delta(q^2)$ part in the propagator while its linear term in q^2 cancels the $1/q^2$ in the other part. This latter contribution will be labeled by the letter ‘X’ in order not to confuse it with the contributions of Fig. 2A.

4.3.2 The non-vanishing contribution

We denote the vertex contribution coming from Fig. 3A by

$$\Gamma^{(X)}_{\lambda\rho}(p,p') \quad = \quad \frac{e^2\gamma^\alpha}{q^2} X_{\lambda\rho\alpha}(q)\,, \tag{4.19}$$

where $X_{\lambda\rho\alpha}(q)$ is the photon-graviton mixing diagram with external momentum q

$$X_{\lambda\rho\alpha}(q) = \sum_f \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[V_{\lambda\rho}(k,k') iS_f(k) i\mathcal{Q}_f\gamma_\alpha iS_f(k')\Big]\,. \tag{4.20}$$

Then, taking the above discussion above into account, the quantity which will appear in the expression for the gravitational mass is given by

$$\widetilde{\Gamma}^{(X)}_{\lambda\rho}(p,p') = \frac{e^2\gamma^\alpha}{q^2}\Big[X_{\lambda\rho\alpha}(q) - X_{\lambda\rho\alpha}(0)\Big] . \tag{4.21}$$

As already mentioned, the sum in Eq. (4.20) is over all species of fermions in the medium, the charge of each species being denoted by $\mathcal{Q}_f$ with the convention that $\mathcal{Q}_e = -1$. The medium-dependent contribution to $X_{\lambda\rho\alpha}(q)$ can be written as

$$X_{\lambda\rho\alpha}(q) = \sum_f \mathcal{Q}_f \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m_f^2)\eta_f(k)\left[\frac{A_{\lambda\rho\alpha}(k,k-q)}{q^2 - 2k\cdot q} + \frac{A_{\lambda\rho\alpha}(k+q,k)}{q^2 + 2k\cdot q}\right] , \tag{4.22}$$

where, for arbitrary 4-momenta k_1 and k_2,

$$\begin{aligned} A_{\lambda\rho\alpha}(k_1,k_2) &= \mathrm{Tr}\Big[V_{\lambda\rho}(k_1,k_2)(\not{k}_1 + m_f)\gamma_\alpha(\not{k}_2 + m_f)\Big] \\ &= \Big[(2k_{1\lambda}k_{1\rho} + k_{1\lambda}k_{2\rho} + k_{2\lambda}k_{1\rho})k_{2\alpha} + (m_f^2 - k_1\cdot k_2)(\eta_{\lambda\alpha}k_{1\rho} + \eta_{\rho\alpha}k_{1\lambda}) \\ &\quad -2\eta_{\lambda\rho}(k_1^2 - m_f^2)k_{2\alpha}\Big] + \Big[k_1 \leftrightarrow k_2\Big] . \end{aligned} \tag{4.23}$$

Putting $k^2 = m_f^2$, we obtain

$$\begin{aligned} A_{\lambda\rho\alpha}(k,k-q) &= [8k_\lambda k_\rho - 4(k_\lambda q_\rho + k_\rho q_\lambda) + 2q_\lambda q_\rho]k_\alpha - [4k_\lambda k_\rho - (k_\lambda q_\rho + k_\rho q_\lambda)]q_\alpha \\ &\quad + k\cdot q[\eta_{\lambda\alpha}(2k-q)_\rho + \eta_{\rho\alpha}(2k-q)_\lambda] - 2\eta_{\lambda\rho}(q^2 - 2k\cdot q)k_\alpha . \end{aligned} \tag{4.24}$$

Since $A_{\lambda\rho\alpha}(k_1,k_2) = A_{\lambda\rho\alpha}(k_2,k_1)$ by definition, $A_{\lambda\rho\alpha}(k+q,k)$ is obtained by changing the sign of q in this expression.

Finally, we mention that the complete one-loop vertex function satisfies the transversality condition, which is implied by the gravitational gauge invariance of the theory. This is shown in Appendix A.

5 Calculation of the gravitational mass

As seen in Eq. (2.54), there are three types of $O(e^2)$ correction to the gravitational mass. One of them is proportional to the inertial mass that was calculated in Sec. 3, and another one involves the wave function renormalization factor derived in Sec. 2.2. In this section we find the contributions from the one-loop vertex diagrams. Since the expressions for those already have an explicit factor of e^2 outside the integral, to evaluate them we can use the tree-level values for the dispersion relation and the spinors associated with the external lepton.

5.1 Terms with the photon distribution from Fig. 2

We first evaluate those terms obtained in Sec. 4 that contain the photon distribution function. In fact, if the temperature of the ambient medium is low ($T \ll m_e$) and the chemical potential of the background electrons is zero, these are the only terms that contribute and they are precisely the ones that were calculated in Ref. [1]. Since we have performed the calculations in a different way, using 1-particle irreducible diagrams only, the following results serve as a good checkpoint between the earlier calculations of Ref. [1] and ours.

5.1.1 Contribution (A1)

From the formula for the gravitational mass given in Eq. (2.54), it follows that we need to calculate the vertex only for $p = p'$, in which case

$$\Gamma'^{(A1)}_{\lambda\rho}(p,p) \;=\; -e^2 \int \frac{d^4k}{(2\pi)^3}\, \frac{\delta(k^2)\eta_\gamma(k)}{4(k\cdot p)^2} \Lambda_{\lambda\rho}(k+p, k+p)\,. \tag{5.1}$$

From Eq. (4.8) it follows that, for any 4-vector y^μ,

$$\Lambda_{\lambda\rho}(y,y) = -4y_\lambda y_\rho(\not{y} - 2m_\ell) + (y^2 - m_\ell^2)\Big[(\gamma_\lambda y_\rho + \gamma_\rho y_\lambda) + 2\eta_{\lambda\rho}(\not{y} - 2m_\ell)\Big]\,, \tag{5.2}$$

which leads to

$$\begin{aligned}\overline{u}_s(p)\Gamma'^{(A1)}_{\lambda\rho}(p,p)u_s(p) \;&=\; -e^2 \int \frac{d^4k}{(2\pi)^3}\, \frac{\delta(k^2)\eta_\gamma(k)}{(k\cdot p)^2} \\ &\times\; \overline{u}_s(p)\left[-\frac{k\cdot p}{m_\ell} k_\lambda p_\rho + m_\ell(k_\lambda k_\rho + p_\lambda p_\rho) + \frac{(k\cdot p)^2}{m_\ell}\eta_{\lambda\rho}\right] u_s(p)\,,\end{aligned} \tag{5.3}$$

where we have used Eq. (2.35) and omitted the terms odd in k, which integrate to zero. Using the notation

$$m'_{(A1)} = (2v^\lambda v^\rho - \eta^{\lambda\rho})\left[\overline{u}_s(p)\Gamma'^{(A1)}_{\lambda\rho}(p,p)u_s(p)\right]_{p^\mu=(m_\ell,\vec{0})}\,, \tag{5.4}$$

we obtain

$$\begin{aligned} m'_{(A1)} \;&=\; e^2 \int \frac{d^3K}{(2\pi)^3}\, f_\gamma(K)\Big[\frac{1}{m_\ell K} - \frac{m_\ell}{K^3}\Big] \\ &=\; \frac{e^2T^2}{12m_\ell} - \frac{e^2 m_\ell}{2\pi^2}\int_0^\infty \frac{dK}{K}\, f_\gamma(K)\,.\end{aligned} \tag{5.5}$$

The remaining integral is infrared divergent, but its contribution to the gravitational mass is canceled by a similar term that arises from the wavefunction renormalization, as we show below.

5.1.2 Contribution (B1)

This term has to be treated carefully because the denominators in the integrand of Eq. (4.14) vanish for $q = 0$. However, a careful evaluation of this term, following the procedure indicated in Eq. (2.54), shows that the limit exists. Denoting

$$m'_{(B1)} \equiv (2v^\lambda v^\rho - \eta^{\lambda\rho}) \lim_{\vec{P}\to 0}\left\{\left[\overline{u}_s(p')\Gamma'^{(B1)}_{\lambda\rho}(p,p')u_s(p)\right]_{\substack{E'_\ell = E_\ell \\ \vec{P}\,'\to\vec{P}}}\right\}\,, \tag{5.6}$$

the result is

$$m'_{(B1)} \;=\; -\frac{e^2T^2}{3m_\ell}\,. \tag{5.7}$$

The details of the derivation of this result are given in Appendix B.

5.1.3 Contributions (C1+D1)

We can proceed as in the evaluation of $m'_{(A1)}$ above. Thus, from Eq. (4.16),

$$\overline{u}_s(p)\Gamma'^{(C1+D1)}_{\lambda\rho}(p,p)u_s(p) = -\,e^2 a_{\mu\nu\lambda\rho}\int \frac{d^4k}{(2\pi)^3}\,\frac{\delta(k^2)\eta_\gamma(k)}{k\cdot p}\left[\overline{u}_s(p)\gamma^\mu \not{k}\gamma^\nu u_s(p)\right], \tag{5.8}$$

using the fact that $a_{\mu\nu\lambda\rho}$ is symmetric in the indices μ,ν. Then using

$$(2v^\lambda v^\rho - \eta^{\lambda\rho})a_{\mu\nu\lambda\rho} = -\eta_{\mu\nu} - 2v_\mu v_\nu \tag{5.9}$$

it follows that

$$\begin{aligned} m'_{(C1+D1)} &\equiv (2v^\lambda v^\rho - \eta^{\lambda\rho})\left[\overline{u}_s(p)\Gamma'^{(C1+D1)}_{\lambda\rho}(p,p)u_s(p)\right]_{p^\mu=(m_\ell,\vec{0})} \\ &= 0\,. \end{aligned} \tag{5.10}$$

5.2 Terms with the electron distribution from Fig. 2

These terms contribute only to the vertex involving electrons and positrons. The integration over k_0 and the angular variables can be done exactly. The remaining integral can be evaluated analytically only for special cases of the distribution functions, some of which we consider afterwards.

5.2.1 Contribution (A2)

As can be seen from Eq. (4.11), the denominators of the integrand of this term vanish as $q \to 0$. Consequently, the prescription indicated in Eq. (2.54) has to be followed carefully in this case. As we show in detail in Appendix B.2, defining

$$m'_{(A2)} \equiv (2v^\lambda v^\rho - \eta^{\lambda\rho})\lim_{\vec{P}\to 0}\left\{\left[\overline{u}_s(p')\Gamma'^{(A2)}_{\lambda\rho}(p,p')u_s(p)\right]_{\substack{E'_\ell=E_\ell\\ \vec{P}'\to\vec{P}}}\right\}, \tag{5.11}$$

the final result for this term is

$$\begin{aligned} m'_{(A2)} = \frac{e^2}{m_e}\int\frac{d^3K}{(2\pi)^3 2E_K}\Bigg\{&\frac{2E_K^2-m_e^2}{E_K}\left(\frac{E_K-2m_e}{E_K-m_e}\,\frac{\partial f_e}{\partial E_K}+\frac{E_K+2m_e}{E_K+m_e}\,\frac{\partial f_{\bar{e}}}{\partial E_K}\right) \\ &+\frac{2E_K^4-E_K^3m_e-5E_K^2m_e^2+2E_Km_e^3-2m_e^4}{m_eE_K^2(E_K-m_e)}f_e \\ &-\frac{2E_K^4+E_K^3m_e-5E_K^2m_e^2-2E_Km_e^3-2m_e^4}{m_eE_K^2(E_K+m_e)}f_{\bar{e}}\Bigg\}, \end{aligned} \tag{5.12}$$

where E_K is defined in Eq. (3.21).

5.2.2 Contribution (B2)

From Eq. (4.15) it is seen that the integrand is not singular in the limit $q \to 0$. Therefore we can evaluate directly

$$\Gamma'^{(B2)}_{\lambda\rho}(p,p) = -e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_e^2)\eta_e(k)\gamma^\nu(\not{k}+m_e)\gamma^\mu\,\frac{C_{\mu\nu\lambda\rho}(p-k,p-k)}{(p-k)^4}, \tag{5.13}$$

and the contribution to the gravitational mass is given by

$$m'_{(B2)} = (2v^\lambda v^\rho - \eta^{\lambda\rho}) \left[\overline{u}_s(p)\Gamma'^{(B2)}_{\lambda\rho}(p,p)u_s(p)\right]_{p^\mu=(m_e,\vec{0})} . \tag{5.14}$$

In the expression for $C_{\mu\nu\lambda\rho}$, any term having a factor of $(p-k)_\mu$ or $(p-k)_\nu$ does not contribute to the integral. This is because, within the spinors, we can write

$$\gamma^\nu(\not{k} + m_e)\gamma^\mu(p-k)_\mu = \gamma^\nu(\not{k} + m_e)(m_e - \not{k}) = \gamma^\nu(m_e^2 - k^2) , \tag{5.15}$$

which vanishes because of the δ-function. The argument is similar for $(p-k)_\nu$. Thus,

$$\begin{aligned}&(2v^\lambda v^\rho - \eta^{\lambda\rho})\overline{u}_s(p)\gamma^\nu(\not{k} + m_e)\gamma^\mu C_{\mu\nu\lambda\rho}(p-k, p-k)u_s(p) \\ &\quad = \overline{u}_s(p)\Big\{8(\not{k} - 2m_e)(p\cdot v - k\cdot v)^2 + 4(m_e - 2k\cdot v\not{v})(p-k)^2\Big\}u_s(p) ,\end{aligned} \tag{5.16}$$

ignoring all terms which have a factor of $k^2 - m_e^2$. Using Eqs. (2.35) and (2.36), we then obtain

$$\begin{aligned} m'_{(B2)} &= -\frac{2e^2}{m_e^2}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_e^2)\eta_e(k)\left(k_0 + \frac{m_e^2}{k_0 - m_e}\right) \\ &= -\frac{e^2}{m_e^2}\int \frac{d^3K}{(2\pi)^3}\,(f_e - f_{\bar{e}}) - \frac{e^2}{2\pi^2}\int dK\,\frac{K^2}{E_K}\left[\frac{f_e(E_K)}{E_K - m_e} - \frac{f_{\bar{e}}(E_K)}{E_K + m_e}\right] . \end{aligned} \tag{5.17}$$

5.2.3 Contributions (C2+D2)

Similarly, for this term we can evaluate directly

$$m'_{(C2+D2)} = (2v^\lambda v^\rho - \eta^{\lambda\rho}) \left[\overline{u}_s(p)\Gamma'^{(C2+D2)}_{\lambda\rho}(p,p)u_s(p)\right]_{p^\mu=(m_e,\vec{0})} , \tag{5.18}$$

with

$$\Gamma'^{(C2+D2)}_{\lambda\rho}(p,p) = 2e^2 a_{\mu\nu\lambda\rho}\int \frac{d^4k}{(2\pi)^3}\delta(k^2-m_e^2)\eta_e(k)\,\frac{\gamma^\mu(\not{k}+m_e)\gamma^\nu}{(k-p)^2} . \tag{5.19}$$

By straight forward algebra

$$(2v^\lambda v^\rho - \eta^{\lambda\rho})a_{\mu\nu\lambda\rho}\gamma^\mu(\not{k}+m_e)\gamma^\nu = 4\not{k} - 6m_e - 4k\cdot v\not{v} , \tag{5.20}$$

and using Eqs. (2.35) and (2.36),

$$\begin{aligned} m'_{(C2+D2)} &= 6e^2\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_e^2)\eta_e(k)\frac{1}{k_0 - m_e} \\ &= \frac{3e^2}{2\pi^2}\int dK\,\frac{K^2}{E_K}\left[\frac{f_e(E_K)}{E_K - m_e} - \frac{f_{\bar{e}}(E_K)}{E_K + m_e}\right]. \end{aligned} \tag{5.21}$$

5.3 Terms from Fig. 3

The contribution to the gravitational mass due to this term is

$$m'_{(X)} \equiv (2v^\lambda v^\rho - \eta^{\lambda\rho}) \lim_{\vec{P}\to 0}\left\{\left[\overline{u}_s(p')\widetilde{\Gamma}^{(X)}_{\lambda\rho}(p,p')u_s(p)\right]_{\substack{q^0=0\\ \vec{Q}\to\vec{0}}}\right\} , \tag{5.22}$$

where, from Eq. (4.21),

$$\widetilde{\Gamma}^{(X)}_{\lambda\rho}(p,p')\Big|_{q^0=0} = -\frac{e^2\gamma^\alpha}{Q^2}\left[X_{\lambda\rho\alpha}(q)\Big|_{q^0=0} - X_{\lambda\rho\alpha}(0)\right] . \tag{5.23}$$

Using the expression for $A_{\lambda\rho\alpha}$ from Eq. (4.24) we obtain

$$\begin{aligned}(2v^\lambda v^\rho - \eta^{\lambda\rho})X_{\lambda\rho\alpha}(q)\Big|_{q^0=0} &= 8\sum_f \mathcal{Q}_f \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m_f^2)\eta_f(k)k_0 v_\alpha \\ &\times \left[(2k_0^2 - m_f^2 - \frac{1}{2}Q^2)\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2} - \frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right)\right]\end{aligned} \tag{5.24}$$

where we have omitted the terms that vanish by symmetric integration over $\vec{K}$, as well as all those terms that are independent of q, because they drop out of Eq. (5.23), and in addition all the terms that are proportional to q_α, because in Eq. (5.22) they yield a factor of $\not{q}$ which vanishes between spinors. Performing the integration over k^0,

$$\begin{aligned}(2v^\lambda v^\rho - \eta^{\lambda\rho})X_{\lambda\rho\alpha}(q)\Big|_{q^0=0} &= 4v_\alpha\sum_f \mathcal{Q}_f \int \frac{d^3K}{(2\pi)^3}\left(f_f - f_{\bar{f}}\right) \\ &\times\ (2E_K^2 - m_f^2 - \frac{1}{2}Q^2)\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2} - \frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right).\end{aligned} \tag{5.25}$$

For the term that contains an explicit factor of Q^2 in the numerator we use the angular integration formula of Eq. (B.9), which yields

$$\begin{aligned}(2v^\lambda v^\rho - \eta^{\lambda\rho})X_{\lambda\rho\alpha}(q)\Big|_{q^0=0} &= v_\alpha\sum_f \mathcal{Q}_f \Bigg[\frac{Q^2}{2\pi^2}\int dK\,\left(f_f - f_{\bar{f}}\right) \\ &+\ 4\int\frac{d^3K}{(2\pi)^3}\left(f_f - f_{\bar{f}}\right)(2E_K^2 - m_f^2)\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2} - \frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right)\Bigg].\end{aligned} \tag{5.26}$$

The evaluation of the rest of the integral is presented in Appendix B.3. Substituting the results into Eq. (5.23), the contribution of this diagram to the gravitational mass is found to be given by

$$m'_{(X)} = -e^2\sum_f \frac{\mathcal{Q}_f}{6\pi^2}\int dK\left[\left(f_f - f_{\bar{f}}\right) - \frac{2E_K^2 - m_f^2}{2E_K}\frac{\partial}{\partial E_K}\left(f_f - f_{\bar{f}}\right)\right]. \tag{5.27}$$

5.4 Summary

Starting from Eq. (2.54), the total contribution to the gravitational mass of charged leptons can be written in the form

$$M'_\ell = m_\ell + m'_{\ell 1} + m'_{\ell 2} + m'_{(X)}\,, \tag{5.28}$$

where $m'_{(X)}$ is the contribution from Eq. (5.27), which is the same for all charged leptons, $m'_{\ell 1}$ represents the terms that contain the photon distribution function, and $m'_{\ell 2}$ contains the terms that depend on the electron distribution function. They are given as follows.

Substituting into Eq. (2.54) the results given in Eqs. (5.5), (5.7) and (5.10), and using the expression for the wave-function normalization and the correction to the inertial mass given in Eqs. (3.28) and (3.16), we find

$$m'_{\ell 1} = -\frac{e^2 T^2}{12 m_\ell}\,, \tag{5.29}$$

in agreement with the DHR result[1], quoted in Eq. (1.4). Notice that the infrared divergence contained in the $m'_{(A1)}$ term cancels with a similar one that arises from the wave function renormalization correction $\zeta_{\ell 1}$.

The terms from the diagrams in Fig. 2 that involve the fermion distribution function contribute only to the gravitational mass of the electron, and therefore

$$m'_{\mu 2} = m'_{\tau 2} = 0\,. \tag{5.30}$$

The individual contributions of this type to the electron gravitational mass appear in Eqs. (5.12), (5.17) and (5.21). Substituting those results into Eq. (2.54), and using the results for the inertial mass and the wave-function normalization factor, given in Eqs. (3.20) and (3.32) respectively, we obtain

$$\begin{aligned} m'_{e2} \;=\; & \frac{e^2}{\pi^2 m_e}\int_0^\infty dK\,\frac{K^2}{2E_K}\left\{\left(\frac{3}{2}+\frac{m_e^2}{E_K^2}-\frac{m_e}{E_K-m_e}\right) f_e(E_K)\right. \\ & +\left(\frac{3}{2}+\frac{m_e^2}{E_K^2}+\frac{m_e}{E_K+m_e}\right) f_{\bar e}(E_K) \\ & \left.+\frac{2E_K^2-m_e^2}{2E_K}\left(\frac{E_K-2m_e}{E_K-m_e}\,\frac{\partial f_e}{\partial E_K}+\frac{E_K+2m_e}{E_K+m_e}\,\frac{\partial f_{\bar e}}{\partial E_K}\right)\right\}. \end{aligned} \tag{5.31}$$

The corresponding formulas for the antileptons are obtained by making the substitution $v^\mu \to -v^\mu$, as indicated in Eq. (2.61). Since the dependence of $\Gamma'_{\lambda\rho}$ on v^μ arises only through the factors η_f and η_γ defined in Eqs. (3.8) and (3.9), the substitution $v^\mu \to -v^\mu$ is equivalent to the prescription

$$\mu_f \to -\mu_f\,, \tag{5.32}$$

or equivalently,

$$f_f \leftrightarrow f_{\bar f}\,. \tag{5.33}$$

Thus,

$$M'_{\bar\ell} = m_\ell + m'_{\ell 1} + m'_{\bar\ell 2} - m'_X\,, \tag{5.34}$$

where

$$m'_{\bar\mu 2} = m'_{\bar\tau 2} = 0\,, \tag{5.35}$$

while the result for $m'_{\bar e 2}$ is obtained from Eq. (5.31) by making the substitution $f_e \leftrightarrow f_{\bar e}$.

6 Results for particular cases

In contrast with the $m_{\ell 1}$ and $m'_{\ell 1}$ terms, which depend on the photon momentum distribution, m_{e2}, $m_{\bar{e}2}$, m'_{e2}, $m'_{\bar{e}2}$ and $m'_{(X)}$ depend on the fermion distribution functions and cannot be evaluated exactly in the general case. Therefore, for illustration, we consider in detail their calculation for the specific situation in which the background is composed of non-relativistic protons and electrons. In this case we can set (for $f = e, p$)

$$f_{\bar{f}}(E) \approx 0 \tag{6.1}$$

and

$$E_K \approx m_f + \frac{K^2}{2m_f} \,. \tag{6.2}$$

We consider in detail two cases separately, according to whether the electron gas is classical or degenerate.

6.1 Classical electron gas and classical proton gas

In this situation we can set

$$f_f(E) = e^{-\beta(E-\mu_f)} \tag{6.3}$$

for both $f = e, p$. This implies the relation

$$\frac{\partial f_f}{\partial E} = -\beta f_f \,, \tag{6.4}$$

as well as the integration formula

$$\int dK\; K^{2r} f_f = 2\pi^{3/2}\Gamma\left(r + \frac{1}{2}\right)\left(\frac{\beta}{2m_f}\right)^{1-r} n_f \,, \tag{6.5}$$

where n_f is the number density, given by

$$n_f = 2\int \frac{d^3K}{(2\pi)^3}\, f_f(E) \approx 2\left(\frac{m_f}{2\pi\beta}\right)^{3/2} e^{-\beta(m_f-\mu_f)} \,. \tag{6.6}$$

Let us consider m_{e2} and $m_{\bar{e}2}$, given in Eqs. (3.20) and (3.23), respectively. Setting $f_{\bar{e}} = 0$ and using Eq. (6.2) to expand the co-efficients of f_e in the integrands in powers of K, the remaining integrals are evaluated by means of Eq. (6.5) to yield

$$\begin{aligned} m_{e2} &= -\frac{e^2 n_e}{2m_e T} + O(n_e/m_e^2)\,, \\ m_{\bar{e}2} &= \frac{3e^2 n_e}{8m_e^2} + O(n_e T^2/m_e^4)\,. \end{aligned} \tag{6.7}$$

Similarly, from Eq. (5.27) we obtain for this case

$$\begin{aligned} m'_{(X)} &= -\frac{7e^2}{24T}\sum_{f=e,p}\frac{\mathcal{Q}_f n_f}{m_f} + O(n_e T/m_e^3) \\ &\approx \frac{7e^2 n_e}{24 m_e T}\,, \end{aligned} \tag{6.8}$$

where we have used the charge-neutrality condition which, neglecting terms $O(\kappa)$, is simply $\sum_f \mathcal{Q}_f n_f = 0$. Applying the same procedure in Eq. (5.31), the leading contribution, in powers of T/m_e, comes from the $\partial f_e/\partial E_K$ term in that equation, and leads to

$$m'_{e2} = \frac{e^2 n_e}{2T^2} + O(n_e/Tm_e)\,. \tag{6.9}$$

By the substitution indicated in Eq. (5.33), the corresponding result for the positron is

$$m'_{\bar{e}2} = -\frac{3e^2 n_e}{8m_e T} + O(n_e/m_e^2)\,. \tag{6.10}$$

Therefore, using Eqs. (3.15), (5.28) and (5.34), the inertial and gravitational masses for charged leptons ℓ other than the electron are obtained as

$$\begin{aligned} M_{\bar{\ell}} = M_\ell &= m_\ell + \frac{e^2T^2}{12m_\ell}\,, \\ M'_{\ell,\bar{\ell}} &= m_\ell - \frac{e^2T^2}{12m_\ell} \pm \frac{7e^2 n_e}{24m_e T}\,, \end{aligned} \tag{6.11}$$

where the upper sign corresponds to the leptons and the lower one to the anti-leptons. The corresponding formulas for the electron are

$$\begin{aligned} M_e &= m_e + \frac{e^2T^2}{12m_e} - \frac{e^2 n_e}{2m_e T}\,, \\ M'_e &= m_e - \frac{e^2T^2}{12m_e} + \frac{e^2 n_e}{2T^2}\,, \end{aligned} \tag{6.12}$$

and for the positron they are

$$\begin{aligned} M_{\bar{e}} &= m_e + \frac{e^2T^2}{12m_e} + \frac{3e^2 n_e}{8m_e^2}\,, \\ M'_{\bar{e}} &= m_e - \frac{e^2T^2}{12m_e} - \frac{2e^2 n_e}{3m_e T}\,. \end{aligned} \tag{6.13}$$

We now estimate how large these corrections could be for the electron. Those due to the photon background were estimated by DHR[1] and were found to be extremely small. Therefore, neglecting that contribution, the fractional change in the inertial and gravitational mass is given by

$$\left|\frac{M_e - m_e}{m_e}\right| = \frac{e^2 n_e}{2m_e^2 T} \tag{6.14}$$

$$\left|\frac{M'_e - m_e}{m_e}\right| = \frac{e^2 n_e}{2m_e T^2}\,. \tag{6.15}$$

Although it may seem that the effects are more noticeable as the temperature decreases, they are bounded by the condition that the electron gas is non-degenerate and non-interacting, which requires that [10]

$$T > \frac{n_e^{2/3}}{m_e} \tag{6.16}$$

and

$$T > \frac{e^2}{r_{\rm av}} \sim e^2 n_e^{1/3}\,, \tag{6.17}$$

since $r_{\rm av} \sim n_e^{-1/3}$. Using the fact that $n_p = n_e$, it follows that the corresponding conditions for the proton gas do not imply further restrictions, because they are automatically satisfied whenever Eqs. (6.16) and (6.17) hold.

By writing the right-hand side of Eq. (6.14) in the alternative forms

$$\begin{aligned} \frac{e^2 n_e}{2m_e^2 T} &= \left(\frac{e^2 n_e^{1/3}}{T}\right)\left(\frac{n_e^{2/3}}{2m_e^2}\right) \\ &= \left(\frac{n_e^{2/3}}{m_e T}\right)^2 \left(\frac{T}{2m_e}\right)\left(\frac{e^2 m_e}{n_e^{1/3}}\right)\,, \end{aligned} \tag{6.18}$$

it is seen that

$$\left|\frac{M_e - m_e}{m_e}\right| < \begin{cases} e^4/2 & \text{if } n_e^{1/3} < e^2 m_e\,, \\ T/2m_e & \text{if } n_e^{1/3} > e^2 m_e\,. \end{cases} \tag{6.19}$$

Similarly, writing

$$\begin{aligned} \frac{e^2 n_e}{2m_e T^2} &= \frac{1}{2}\left(\frac{n_e^{1/3}}{e^2 m_e}\right)\left(\frac{e^2 n_e^{1/3}}{T}\right)^2 \\ &= \frac{1}{2}\left(\frac{n_e^{2/3}}{m_e T}\right)^2 \left(\frac{e^2 m_e}{n_e^{1/3}}\right)\,, \end{aligned} \tag{6.20}$$

it follows that

$$\left|\frac{M_e' - m_e}{m_e}\right| < \frac{1}{2} \tag{6.21}$$

in either case. Therefore, while the fractional correction to the electron's inertial mass is likely to be small in most situations with the conditions that we are presently considering, the fractional change in the gravitational mass could be substantial. For example, if we use the temperature and density at the solar core, i.e., $T = 1.57 \times 10^7$ K, $n_e = 9.5 \times 10^{25}$ cm^{-3}, we obtain

$$\begin{aligned} \left|\frac{M_e - m_e}{m_e}\right| &= 9.8 \times 10^{-5}\,, \\ \left|\frac{M_e' - m_e}{m_e}\right| &= 3.5 \times 10^{-2}\,, \end{aligned} \tag{6.22}$$

which shows that the correction to the gravitational mass of the electron can at least be appreciable in realistic physical situations.

6.2 Degenerate electron gas and classical proton gas

In this case

$$T \ll \frac{n_e^{2/3}}{m_e} \sim \frac{K_F^2}{m_e}\,, \tag{6.23}$$

where K_F is the Fermi momentum of the electron gas. We assume

$$K_F \ll m_e \tag{6.24}$$

so that the electrons are non-relativistic, and

$$K_F > e^2 m_e \,, \tag{6.25}$$

which implies that the average kinetic energy of an electron is larger than the average Coulomb interaction energy $\sim e^2 n_e^{1/3} \sim e^2 K_F$, and therefore the electron gas can be treated as an ideal gas. Under these conditions, the protons can be treated as a weakly coupled Boltzmann gas if we assume that the weak coupling condition

$$T \gg e^2 n_p^{1/3} \sim e^2 K_F \tag{6.26}$$

is satisfied[11]. Remembering that $K_F \ll m_e \ll m_p$, this in turn implies the non-degeneracy condition

$$T \gg \frac{K_F^2}{m_p} \sim \frac{n_p^{2/3}}{m_p} \,. \tag{6.27}$$

Therefore, Eq. (6.3) applies to the proton, while for the electron

$$f_e = \Theta(K_F - K) \tag{6.28}$$

with

$$K_F = (3\pi^2 n_e)^{1/3} \,, \tag{6.29}$$

which in turn imply the relation

$$\frac{df_e}{dK} = -\delta(K_F - K) \,. \tag{6.30}$$

We repeat the calculation of the quantities m_{e2}, $m_{\bar{e}2}$, m'_{e2}, $m'_{\bar{e}2}$ and m'_X for this case, neglecting the terms that are a factor $\sim O(K_F^2/m_e^2)$ smaller than the ones that we retain. From Eqs. (3.20) and (3.23), setting $f_{\bar{e}} = 0$ and using Eq. (6.2), we obtain

$$\begin{aligned} m_{e2} &= -\frac{e^2 K_F}{2\pi^2} \,, \\ m_{\bar{e}2} &= \frac{e^2 K_F^3}{8\pi^2 m_e^2} \,. \end{aligned} \tag{6.31}$$

From Eq. (5.27),

$$m'_{(X)} = \frac{e^2}{6\pi^2} \int dK \left[f_e - \left(K + \frac{m_e^2}{2K} \right) \frac{df_e}{dK} \right] - \frac{7e^2}{24T} \frac{n_p}{m_p} \,, \tag{6.32}$$

where we have borrowed the result for the proton contribution from Eq. (6.8), while in the electron term we have expressed E_K in terms of K and used

$$\frac{d}{dE_K} = \frac{E_K}{K} \frac{d}{dK} \tag{6.33}$$

for any function of E_K. Using Eqs. (6.28) and (6.30) this finally yields

$$m'_{(X)} = \frac{e^2 m_e^2}{12\pi^2 K_F}\,. \tag{6.34}$$

Here we have neglected the proton contribution because it is $\sim e^2 K_F^3/(Tm_p) \ll e^2 K_F$ from Eq. (6.27). In a similar fashion, from Eq. (5.31),

$$m'_{e2} = \frac{e^2 m_e^2}{2\pi^2 K_F}\,, \tag{6.35}$$

and by the substitution indicated in Eq. (5.33), the corresponding result for the positron is

$$m'_{\bar{e}2} = -\frac{3e^2 K_F}{8\pi^2}\,. \tag{6.36}$$

Thus, substituting these results into Eqs. (3.15), (5.28) and (5.34), we obtain the following expressions for the inertial and gravitational masses, retaining only the leading terms in powers of K_F/m_e. For the charged leptons ℓ other than the electron,

$$\begin{aligned} M_{\bar{\ell}} = M_\ell &= m_\ell + \frac{e^2 T^2}{12 m_\ell} \\ M'_{\ell,\bar{\ell}} &= m_\ell - \frac{e^2 T^2}{12 m_\ell} \pm \frac{e^2 m_e^2}{12\pi^2 K_F}\,, \end{aligned} \tag{6.37}$$

with the upper sign corresponding to the leptons and the lower one to the anti-leptons, while for the electron

$$\begin{aligned} M_e &= m_e + \frac{e^2 T^2}{12 m_e} - \frac{e^2 K_F}{2\pi^2}\,, \\ M'_e &= m_e - \frac{e^2 T^2}{12 m_e} + \frac{7 e^2 m_e^2}{12\pi^2 K_F}\,, \end{aligned} \tag{6.38}$$

and for the positron

$$\begin{aligned} M_{\bar{e}} &= m_e + \frac{e^2 T^2}{12 m_e} + \frac{e^2 K_F^3}{8\pi^2 m_e^2}\,, \\ M'_{\bar{e}} &= m_e - \frac{e^2 T^2}{12 m_e} - \frac{e^2 m_e^2}{12\pi^2 K_F}\,. \end{aligned} \tag{6.39}$$

It is interesting to note that Eqs. (6.23) and (6.24) imply that the photon contributions in Eqs. (6.37)-(6.39) are much smaller than the contribution due to the electron background in each case. In fact, using Eq. (6.25), we see that the fractinal corrections to the gravitational mass can be as large as about $7/12\pi^2$ for the electron and $1/12\pi^2$ for the positron and the other leptons.

7 Conclusions

In this work we derived a general operational formula that expresses the gravitational mass of a fermion in terms of the gravitational vertex function. Using that formula as the staring point, we have studied the $O(e^2)$ corrections to the gravitational interactions of a charged lepton in the presence of a matter background. This calculation extends and complements previous calculations along similar lines, in various useful ways.

From a technical point of view, the calculations that we have presented have employed various finite-temperature-field-theory techniques that can be useful also in other contexts. For example, we have shown in detail how a careful treatment of the wavefunction renormalization factor, which arises from considering the one-particle reducible diagrams in the proper way, is instrumental in the cancelation of an infrared divergent contribution that arises from the photon contribution to the proper vertex function.

On the other hand, a well known problem that arises in this type of calculation is the ambiguity of the finite temperature Green functions when they are evaluated at zero momentum[7]. This property is usually due to the fact that the different mathematical limits correspond to different physical situations, so in those cases the resolution of the apparent paradox lies in recognizing the appropriate correspondence with the physical situation at hand [12]. The calculations that we have presented have illustrated this in a particularly convincing way. The operational formula for the gravitational mass given, in Eq. (2.54), indicate the precise order in which the various limits must be taken, according to the physical situation that we considered. As we have shown, the careful application of that prescription has allowed us to evaluate all the integrals involved, in a unique and well-defined way, including those that superficially seem to be singular, without having to introduce by hand any special regularization technique.

The calculations and the results are also important from a phenomenological point of view. As we have indicated, in a matter background with a non-zero chemical potential such as the Sun, the matter contributions to the gravitational mass are proportional to the electron and nucleon densities and its magnitude can be appreciable. These matter contributions dominate over the photon-background contribution, especially in those situations in which $T \ll m_e$, for which the photon contribution becomes negligible. Moreover, the matter-induced corrections to the gravitational mass are different for the various charged lepton flavors, and are not be the same for the corresponding antiparticles. There are situations in which mass differences, intrinsic or induced, have important physical implications, such as the neutron-proton mass difference in the context of the nucleosynthesis calculations in the Early Universe. Although our work has focused in the case of the charged leptons, similar considerations can be applied to the other fermions as well. The question of the possible implications of this type of mass correction in specific situations is an important one, but is outside the scope of the present work. Nevertheless, our calculations have provided a necessary ingredient for being able to consider them in a systematic manner, and set the stage for their further study on a firm basis.

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Appendices

A Transversality of the vertex

It is useful to verify that the complete vertex function to $O(e^2)$, obtained in Sec. 4, satisfies the transversality conditions

$$q^\lambda \overline{U}_s(p')\,\Gamma_{\lambda\rho}(p,p')\,U_s(p) = q^\rho \overline{U}_s(p')\,\Gamma_{\lambda\rho}(p,p')\,U_s(p) = 0 \tag{A.1}$$

to this order. Since the vertex is symmetric in the Lorentz indices λ, ρ, either of these relations guarantees the other. In order to simplify the notation, in the remainder of this appendix we omit the subscript s in the spinors.

In order to verify this relation, the important point is that we must include all the terms upto $O(e^2)$. Since the one-loop terms in the induced vertex are already $O(e^2)$, for them we can adopt the tree-level definition of the spinors, i.e.

$$\not p\, u(p) = mu(p)\,, \qquad \overline{u}(p')\not p' = m\overline{u}(p)\,, \tag{A.2}$$

as well as the tree-level on-shell conditions

$$p^2 = p'^2 = m^2\,. \tag{A.3}$$

In the appendices, we use the tree level mass m without any subscript, implying m_ℓ, m_e or m_f which should be understood from the context. Also note that the photon distribution function as well as the associated δ-function are even in k, and therefore those terms which are odd in k in the rest of the integrand do not contribute.

We first show that the vertex contribution from Fig. 3A is transverse by itself. From Eq. (4.24),

$$q^\lambda A_{\lambda\rho\alpha}(k, k-q) = -(q^2 - 2k\cdot q)(4k_\rho k_\alpha + k\cdot q\eta_{\rho\alpha})\,, \tag{A.4}$$

where we omit the terms that are proportional to q_α because, in Eq. (4.19), they will yield $\not q$ which vanishes between the spinors. Changing the sign of q in the last equation yields

$$q^\lambda A_{\lambda\rho\alpha}(k+q, k) = (q^2 + 2k\cdot q)(4k_\rho k_\alpha - k\cdot q\eta_{\rho\alpha})\,, \tag{A.5}$$

and as a result $q^\lambda\overline{u}(p')\Gamma^{(X)}_{\lambda\rho}(p,p')u(p)$ turns out to be proportional to $\sum_f Q_f(n_f - n_{\bar f})$, which is zero to this order.

As for the other diagrams, straightforward algebra gives the following results:

$$\begin{aligned}
q^\lambda\overline{u}(p')\Gamma'^{(A1)}_{\lambda\rho}(p,p')u(p) &= -\frac{e^2}{4}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\\
&\times\ \overline{u}(p')\left[\frac{4mk_\rho - (p+5p')_\rho\not k + 2k\cdot p\gamma_\rho}{k\cdot p'} - \Big(p\leftrightarrow p'\Big)\right]u(p)\\
q^\lambda\overline{u}(p')\Gamma'^{(A2)}_{\lambda\rho}(p,p')u(p) &= \frac{e^2}{4}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\
&\times\ \overline{u}(p')\left[\frac{4(\not k - 2m)k_\rho + \not k(p+p')_\rho - 2k\cdot p'\gamma_\rho}{m^2 - k\cdot p} - \Big(p\leftrightarrow p'\Big)\right]u(p)\,,\\
q^\lambda\overline{u}(p')\Gamma'^{(B1)}_{\lambda\rho}(p,p')u(p) &= e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\\
&\times\ \overline{u}(p')\left[\frac{mk_\rho - \not k p'_\rho}{k\cdot p'} - \frac{mk_\rho - \not k p_\rho}{k\cdot p}\right]u(p)\\
q^\lambda\overline{u}(p')\Gamma'^{(B2)}_{\lambda\rho}(p,p')u(p) &= -e^2\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\
&\times\ \overline{u}(p')\left[\frac{(\not k - 2m)k_\rho + mp_\rho}{m^2 - k\cdot p} - \frac{(\not k - 2m)k_\rho + mp'_\rho}{m^2 - k\cdot p'}\right]u(p)\,,\\
q^\lambda\overline{u}(p')\Gamma'^{(C1+D1)}_{\lambda\rho}(p,p')u(p) &= \frac{e^2}{2}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\left(\frac{1}{k\cdot p'} + \frac{1}{k\cdot p}\right)\\
&\times\ \overline{u}(p')\Big[\not k q_\rho + k\cdot q\gamma_\rho\Big]u(p)\,,
\end{aligned}$$

$$q^\lambda \overline{u}(p')\Gamma'^{(C2+D2)}_{\lambda\rho}(p,p')u(p) = -\frac{e^2}{2}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\left(\frac{1}{m^2-k\cdot p'}+\frac{1}{m^2-k\cdot p}\right)$$
$$\times\ \overline{u}(p')\Big[(\not k-3m)q_\rho + k\cdot q\gamma_\rho\Big]u(p)\,. \tag{A.6}$$

Therefore, adding all the one-loop contributions to the vertex, we obtain

$$q^\lambda \overline{u}(p')\Gamma'^{(1)}_{\lambda\rho}(p,p')u(p) = \frac{e^2}{4}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)$$
$$\times\ \overline{u}(p')\left[\frac{\not k}{k\cdot p'}\Big(3p_\rho - p'_\rho\Big) - \Big(p\leftrightarrow p'\Big)\right]u(p)\,, \tag{A.7}$$
$$q^\lambda \overline{u}(p')\Gamma'^{(2)}_{\lambda\rho}(p,p')u(p) = \frac{e^2}{4}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)$$
$$\times\ \overline{u}(p')\left[\frac{(\not k-2m)(3p'_\rho-p_\rho)-2k\cdot p\gamma_\rho}{m^2-k\cdot p} - \Big(p\leftrightarrow p'\Big)\right]u(p)\,. \tag{A.8}$$

We need to add to these the tree-level contribution to the gravitational vertex that appears in Eq. (2.41). In this case, we must include the $O(e^2)$ corrections to the equation for the spinors, which arise from the self-energy diagrams of Sec. 3. Thus, for this part, using Eq. (2.17) and its hermitian conjugate

$$\overline{U}(p')\Big(\not p' - m - \Sigma(p')\Big) = 0\,, \tag{A.9}$$

we obtain

$$\overline{U}(p')\not q U(p) = \overline{U}(p')\Big(\Sigma'(p)-\Sigma'(p')\Big)U(p)\,, \tag{A.10}$$

which in turn yields

$$q^\lambda \overline{U}(p')\,V_{\lambda\rho}(p,p')\,U(p) = \frac{1}{4}\overline{U}(p')\Big[(3p'-p)_\rho\Sigma'(p) + p^2\gamma_\rho - \Big(p\leftrightarrow p'\Big)\Big]U(p)\,. \tag{A.11}$$

This can be cast in a different form by multiplying Eq. (2.17) from the left by $\overline{U}(p')\gamma_\rho(\not p+m)$ and Eq. (A.9) from the right by $(\not p'+m)\gamma_\rho U(p)$ and taking the difference of the resulting equations. This gives

$$(p^2-p'^2)\,\overline{U}(p')\gamma_\rho U(p) = \overline{U}(p')\Big[\gamma_\rho(\not p+m)\Sigma'(p) - \Sigma'(p')(\not p'+m)\gamma_\rho\Big]U(p)\,, \tag{A.12}$$

and substituting this result into Eq. (A.11), we obtain

$$q^\lambda \overline{U}(p')\,V_{\lambda\rho}(p,p')\,U(p) = \frac{1}{4}\overline{U}(p')\Big[(3p'-p)_\rho\Sigma'(p) - (3p-p')_\rho\Sigma'(p')$$
$$+\gamma_\rho(\not p+m)\Sigma'(p) - \Sigma'(p')(\not p'+m)\gamma_\rho\Big]U(p)\,. \tag{A.13}$$

Since Σ' is explicitly of $O(e^2)$ while we are interested in results to $O(e^2)$ only, we can use the tree-level spinors on the right-hand side. Using Eq. (A.3) in Eq. (3.11), we can write the self-energy contribution involving the photon distribution function as

$$\Sigma'_1(p) = e^2\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\,\frac{\not k}{k\cdot p}\,, \tag{A.14}$$

disregarding terms odd in k. Similarly, from Eq. (3.11), the part containing the Fermi distribution function can be written as

$$\Sigma_2'(p) = -e^2 \int \frac{d^4k}{(2\pi)^3}\, \delta(k^2 - m^2)\eta_F(k)\, \frac{\not{k} - 2m}{m^2 - k\cdot p}\,. \tag{A.15}$$

Substituting these forms into Eq. (A.13) and using the identities

$$\begin{aligned}\overline{u}(p')\gamma_\rho(\not{p}+m)\not{k}u(p) &= 2k\cdot p\,\overline{u}(p')\gamma_\rho u(p)\\ \overline{u}(p')\not{k}(\not{p}'+m)\gamma_\rho u(p) &= 2k\cdot p'\,\overline{u}(p')\gamma_\rho u(p)\,,\end{aligned} \tag{A.16}$$

we see that Eq. (A.13) cancels the contribution from the loop diagrams given in Eqs. (A.7) and (A.8) to this order. This proves the transversality of the effective vertex.

B Apparently singular contributions

B.1 The B1 contribution

We start from the formula given in Eq. (4.14), from which it follows that

$$\begin{aligned}\Gamma'^{(B1)}_{\lambda\rho}(p,p) &= -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\delta(k^2)\eta_\gamma(k)\\ &\times \left[\frac{\gamma^\nu(\not{p}-\not{k}+m)\gamma^\mu C_{\mu\nu\lambda\rho}(k,k-q)}{k\cdot p(2\vec{K}\cdot\vec{Q}-Q^2)} - \frac{\gamma^\nu(\not{p}-\not{k}-\not{q}+m)\gamma^\mu C_{\mu\nu\lambda\rho}(k+q,k)}{(k\cdot p+\vec{K}\cdot\vec{Q})(2\vec{K}\cdot\vec{Q}+Q^2)}\right]\end{aligned} \tag{B.1}$$

where we have put

$$q^\mu = (0,\vec{Q}) \qquad k^\mu = (k^0,\vec{K})\,. \tag{B.2}$$

In order to take the limit $\vec{Q} \to 0$, our strategy is to expand the coefficients of the factors $1/(2\vec{K}\cdot\vec{Q} \pm Q^2)$ in powers of $\vec{Q}$. Of the resulting terms in the coefficients, those which are quadratic in $\vec{Q}$ do not contribute in the $\vec{Q} \to 0$ limit and therefore we need to keep only the terms that are at most linear in $\vec{Q}$.

Using the property $C_{\mu\nu\lambda\rho}(k+q,k) = C_{\nu\mu\lambda\rho}(k,k+q)$, we can write

$$\begin{aligned}C_{\mu\nu\lambda\rho}(k,k-q) &= C_{\mu\nu\lambda\rho}(k,k) - C'_{\mu\nu\lambda\rho}(k,q)\\ C_{\mu\nu\lambda\rho}(k+q,k) &= C_{\mu\nu\lambda\rho}(k,k) + C'_{\nu\mu\lambda\rho}(k,q)\,,\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}C'_{\mu\nu\lambda\rho}(k,q) &= \eta_{\lambda\rho}(\eta_{\mu\nu}k\cdot q - q_\mu k_\nu) - \eta_{\mu\nu}(k_\lambda q_\rho + q_\lambda k_\rho)\\ &\quad + k_\nu(\eta_{\lambda\mu}q_\rho + \eta_{\rho\mu}q_\lambda) + q_\mu(\eta_{\lambda\nu}k_\rho + \eta_{\rho\nu}k_\lambda) - k\cdot q(\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu})\,.\end{aligned} \tag{B.4}$$

To first order in Q, we can also put

$$\frac{1}{k\cdot p + \vec{K}\cdot\vec{Q}} = \frac{1}{k\cdot p} - \frac{\vec{K}\cdot\vec{Q}}{(k\cdot p)^2}\,. \tag{B.5}$$

This enables us to decompose $\Gamma'^{(B1)}_{\lambda\rho}(p,p)$ in the following four terms:

$$
\begin{aligned}
\Gamma'^{(B1a)}_{\lambda\rho}(p) &= -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\frac{C_{\mu\nu\lambda\rho}(k,k)\gamma^\nu(\not{p}-\not{k}+m)\gamma^\mu}{k\cdot p} \\
&\quad\times\left[\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right] \\
\Gamma'^{(B1b)}_{\lambda\rho}(p) &= -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\delta(k^2)\eta_\gamma(k)\frac{C_{\mu\nu\lambda\rho}(k,k)\gamma^\nu\not{q}\gamma^\mu}{k\cdot p(2\vec{K}\cdot Q+Q^2)} \\
\Gamma'^{(B1c)}_{\lambda\rho}(p) &= -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2)\eta_\gamma(k)\frac{C_{\mu\nu\lambda\rho}(k,k)\gamma^\nu(\not{p}-\not{k}+m)\gamma^\mu}{(k\cdot p)^2}\left[\frac{\vec{K}\cdot\vec{Q}}{2\vec{K}\cdot\vec{Q}+Q^2}\right] \\
\Gamma'^{(B1d)}_{\lambda\rho}(p) &= -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\delta(k^2)\eta_\gamma(k)\frac{\gamma^\nu(\not{p}-\not{k}+m)\gamma^\mu}{k\cdot p} \\
&\quad\times\left[-\frac{C'_{\mu\nu\lambda\rho}(k,q)}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{C'_{\nu\mu\lambda\rho}(k,q)}{2\vec{K}\cdot\vec{Q}+Q^2}\right]. \qquad \text{(B.6)}
\end{aligned}
$$

We carry out these integrals one by one.

Eliminating the manifestly k-odd terms from the integrand and performing the k_0-integration, we obtain

$$\Gamma'^{(B1a)}_{\lambda\rho}(p) = e^2\lim_{\vec{Q}\to 0}\int \frac{d^3K}{(2\pi)^3 2K}f_\gamma(K)\frac{4k_\lambda k_\rho\not{k}}{k\cdot p}\left[\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right], \qquad \text{(B.7)}$$

using $k^2=0$. The expression within the square brackets is finite for $Q\to 0$. Therefore, in the spinors we can set $p=p'$, and using Eq. (2.35) we then obtain

$$m'_{(B1a)} = \frac{4e^2}{m}\lim_{\vec{Q}\to 0}\int \frac{d^3K}{(2\pi)^3}f_\gamma(K)K\left[\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right]. \qquad \text{(B.8)}$$

We can perform the integration over the angular variables in $\vec{K}$, the integral being understood, as usual, in terms of the principal value part. That gives

$$\int d\Omega\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2} = -\int d\Omega\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2} = -\frac{\pi}{K^2}+\mathcal{O}(Q^2), \qquad \text{(B.9)}$$

so that

$$m'_{(B1a)} = -\frac{e^2}{\pi^2 m}\int dK\ f_\gamma(K)K \quad = \quad -\frac{e^2T^2}{6m}. \qquad \text{(B.10)}$$

As for the next contribution, it is straightforward to verify that

$$(2v^\lambda v^\rho-\eta^{\lambda\rho})\gamma^\nu\not{q}\gamma^\mu C_{\mu\nu\lambda\rho}(k,k) = 4\vec{K}\cdot\vec{Q}(\not{k}-2k\cdot v\not{v}), \qquad \text{(B.11)}$$

using $q\cdot v=q_0=0$. So

$$(2v^\lambda v^\rho-\eta^{\lambda\rho})\Gamma'^{(B1b)}_{\lambda\rho}(p) \quad = \quad e^2\int \frac{d^4k}{(2\pi)^3}\delta(k^2)\eta_\gamma(k)\frac{(-\not{k}+2k\cdot v\not{v})}{k\cdot p}. \qquad \text{(B.12)}$$

Now using Eqs. (2.35) and (2.36), carrying out the integral over k^0, and finally putting $P=0$, we get

$$m'_{(B1b)} = \frac{2e^2}{m}\int \frac{d^3K}{(2\pi)^3 2K} f_\gamma(K) = \frac{e^2T^2}{12m}\,. \tag{B.13}$$

Similarly,

$$\Gamma'^{(B1c)}_{\lambda\rho}(p) \;=\; -\frac{e^2}{2}\int \frac{d^3K}{(2\pi)^3 2K}\, f_\gamma(K)\gamma^\nu(\not{p}+m)\gamma^\mu C_{\mu\nu\lambda\rho}(k,k)\frac{1}{(k\cdot p)^2}\,, \tag{B.14}$$

and

$$\begin{aligned}(2v^\lambda v^\rho-\eta^{\lambda\rho})\Gamma'^{(B1c)}_{\lambda\rho}(p) \;&=\; -\frac{2e^2}{m}\int \frac{d^3K}{(2\pi)^3 2K}\, f_\gamma(K)\frac{1}{(k\cdot p)^2} \\ &\quad\times\left[-(k\cdot p)^2-2m^2(k\cdot v)^2+4(k\cdot p)(k\cdot v)(p\cdot v)\right]\end{aligned} \tag{B.15}$$

so that

$$m'_{(B1c)} = -\frac{2e^2}{m}\int \frac{d^3K}{(2\pi)^3 2K}\, f_\gamma(K) = -\frac{e^2T^2}{12m}\,. \tag{B.16}$$

For $\Gamma'^{(B1d)}_{\lambda\rho}$ we first perform the integral over k^0. Remembering that in the remaining integral we can change $\vec{K}$ to $-\vec{K}$ and using the fact that $C'_{\mu\nu\lambda\rho}(-k,q) = -C'_{\mu\nu\lambda\rho}(k,q)$, we obtain

$$\begin{aligned}\Gamma'^{(B1d)}_{\lambda\rho}(p) \;&=\; \frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^3K}{(2\pi)^3 2K} f_\gamma(K)\left(\frac{1}{k\cdot p}\right) \\ &\quad\times\Bigg\{\gamma^\nu(\not{p}+m)\gamma^\mu\left[\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right]\left[C'_{\mu\nu\lambda\rho}(k,q)-C'_{\nu\mu\lambda\rho}(k,q)\right] \\ &\quad+\gamma^\nu(-\not{k})\gamma^\mu\left[\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}+\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right]\left[C'_{\mu\nu\lambda\rho}(k,q)+C'_{\nu\mu\lambda\rho}(k,q)\right]\Bigg\} \\ &=\; -\frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^3K}{(2\pi)^3 2K} f_\gamma(K)\frac{1}{k\cdot p}\left\{\frac{\gamma^\nu\not{k}\gamma^\mu}{\vec{K}\cdot\vec{Q}}\left[C'_{\mu\nu\lambda\rho}(k,q)+C'_{\nu\mu\lambda\rho}(k,q)\right]+O(Q)\right\}.\end{aligned} \tag{B.17}$$

We now use

$$(2v^\lambda v^\rho-\eta^{\lambda\rho})\gamma^\nu\not{k}\gamma^\mu\left[C'_{\mu\nu\lambda\rho}(k,q)+C'_{\nu\mu\lambda\rho}(k,q)\right] = 8(k\cdot v)(\vec{K}\cdot\vec{Q})\not{v}\,. \tag{B.18}$$

Then, using Eq. (2.35) and putting $\vec{P}=0$, we get

$$m'_{(B2d)} = -\frac{4e^2}{m}\int \frac{d^3K}{(2\pi)^3 2K} f_\gamma(K) = -\frac{e^2T^2}{6m}\,. \tag{B.19}$$

Adding the results given in Eqs. (B.10), (B.13), (B.16) and (B.19), we get the total contribution from the B1 term presented in Eq. (5.7).

B.2 The A2 contribution

For this contribution, we start from Eq. (4.11). Using Eq. (B.2), we can write it as

$$\begin{aligned}\Gamma'^{(A2)}_{\lambda\rho}(p,p) &= \frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\ &\times\left[\frac{\Lambda_{\lambda\rho}(k,k-q)}{(2\vec{K}\cdot\vec{Q}-Q^2)(m^2-k\cdot p)}-\frac{\Lambda_{\lambda\rho}(k+q,k)}{(2\vec{K}\cdot\vec{Q}+Q^2)(m^2-k\cdot p')}\right]. \qquad \text{(B.20)}\end{aligned}$$

Following the strategy stated below Eq. (B.2), let us now write

$$\begin{aligned}\Lambda_{\lambda\rho}(k,k-q) &= \Lambda_{\lambda\rho}(k,k)+\Lambda'_{\lambda\rho}(k,q)\,,\\ \Lambda_{\lambda\rho}(k+q,k) &= \Lambda_{\lambda\rho}(k,k)+\Lambda''_{\lambda\rho}(k,q)\,, \qquad \text{(B.21)}\end{aligned}$$

and, in the denominator, expand $m^2-k\cdot p'$ in powers of $\vec{Q}$:

$$\frac{1}{m^2-k\cdot p'} = \frac{1}{m^2-k\cdot p}+\frac{\vec{K}\cdot\vec{Q}}{(m^2-k\cdot p)^2}+O(Q^2)\,. \qquad \text{(B.22)}$$

Then we can decompose $\Gamma'^{(A2)}_{\lambda\rho}(p,p)$ into the following terms, omitting higher powers of Q which anyway will not contribute:

$$\begin{aligned}\Gamma'^{(A2a)}_{\lambda\rho}(p,p) &= \frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\ &\times\frac{\Lambda_{\lambda\rho}(k,k)}{m^2-k\cdot p}\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right),\\ \Gamma'^{(A2b)}_{\lambda\rho}(p,p) &= -\frac{e^2}{4}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\frac{\Lambda_{\lambda\rho}(k,k)}{(m^2-k\cdot p)^2}\,,\\ \Gamma'^{(A2c)}_{\lambda\rho}(p,p) &= \frac{e^2}{2}\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\ &\times\frac{1}{m^2-k\cdot p}\left(\frac{\Lambda'_{\lambda\rho}(k,q)}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{\Lambda''_{\lambda\rho}(k,q)}{2\vec{K}\cdot\vec{Q}+Q^2}\right), \qquad \text{(B.23)}\end{aligned}$$

We discuss these contributions one by one.

The A2a contribution

Using Eq. (5.2) and the δ-function appearing in the integrand, we can write

$$\begin{aligned}\Gamma'^{(A2a)}_{\lambda\rho}(p,p) &= -2e^2\lim_{\vec{Q}\to 0}\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\eta_F(k)\\ &\times\frac{k_\lambda k_\rho(\not{k}-2m)}{m^2-k\cdot p}\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right). \qquad \text{(B.24)}\end{aligned}$$

As argued before Eq. (B.8), we can put $\vec{Q}=0$ in the spinors, and use Eq. (2.35). Performing the k_0-integration, we obtain

$$(2v^\lambda v^\rho-\eta^{\lambda\rho})\Gamma^{(A2a)}_{\lambda\rho}(p) = \frac{2e^2}{m}\lim_{\vec{Q}\to 0}\int\frac{d^3K}{(2\pi)^3}\,F(\vec{K})\left(\frac{1}{2\vec{K}\cdot\vec{Q}-Q^2}-\frac{1}{2\vec{K}\cdot\vec{Q}+Q^2}\right), \qquad \text{(B.25)}$$

where the expression on the left is understood to equal the one on the right only between the spinors, and

$$F(\vec{K}) = \frac{2E_K^2 - m^2}{2E_K}\left[\left(1 + \frac{m^2}{m^2 - k\cdot p}\right) f_e + \left(1 + \frac{m^2}{m^2 + k\cdot p}\right) f_{\bar{e}}\right], \tag{B.26}$$

with $k_0 = E_K$. Since the integrand contains $\vec{K}\cdot\vec{P}$, and we must set $\vec{P} = 0$ only after taking the limit $Q \to 0$, the angular integrations cannot be performed using Eq. (B.9). So we shift the integration variable to $\vec{K} \pm \frac{1}{2}\vec{Q}$ in the terms having $2\vec{K}\cdot\vec{Q} \mp Q^2$ in the denominator. This gives

$$(2v^\lambda v^\rho - \eta^{\lambda\rho})\Gamma^{(A2a)}_{\lambda\rho}(p) \;=\; \frac{2e^2}{m} \lim_{\vec{Q}\to 0} \int \frac{d^3K}{(2\pi)^3} \frac{\vec{Q}\cdot\vec{\nabla}_K F}{2\vec{K}\cdot\vec{Q}}. \tag{B.27}$$

Clearly the magnitude of $\vec{Q}$ now cancels out. The derivative with respect to $\vec{K}$ can be taken easily, using

$$\begin{aligned} \vec{\nabla}_K E_K &= \frac{\vec{K}}{E_K}, \\ \vec{\nabla}_K \left(\frac{1}{m^2 \pm k\cdot p}\right) &= \frac{\mp 1}{(m^2 \pm k\cdot p)^2}\left(E_P \frac{\vec{K}}{E_K} - \vec{P}\right). \end{aligned} \tag{B.28}$$

The term proportional to $\vec{P}$ from the last derivative does not contribute because it multiplies a factor whose integrand is odd in $\vec{K}$ at $\vec{P} = 0$. Putting $\vec{P} = 0$ in the other terms, we obtain the contribution to the gravitational mass:

$$\begin{aligned} m'_{(A2a)} = \frac{2e^2}{m}\int \frac{d^3K}{(2\pi)^3 2E_K} \times & \left\{ \frac{2E_K^2 - m^2}{2E_K}\left(\frac{E_K - 2m}{E_K - m}\frac{\partial f_e}{\partial E_K} + \frac{E_K + 2m}{E_K + m}\frac{\partial f_{\bar{e}}}{\partial E_K}\right)\right. \\ & + \frac{2E_K^2 - m^2}{2E_K}\left(\frac{m}{(E_K - m)^2} f_e - \frac{m}{(E_K + m)^2} f_{\bar{e}}\right) \\ & \left. + \frac{2E_K^2 + m^2}{2E_K^2}\left(\frac{E_K - 2m}{E_K - m} f_e + \frac{E_K + 2m}{E_K + m} f_{\bar{e}}\right)\right\}. \end{aligned} \tag{B.29}$$

The A2b contribution

The integral in the (A2b) term is independent of Q. So, in a straight forward way, we obtain

$$m'_{(A2b)} \;=\; \frac{e^2}{m^2}\int \frac{d^3K}{(2\pi)^3 2E_K}(2E_K^2 - m^2)\left[\frac{E_K - 2m}{(E_K - m)^2} f_e(E_K) - \frac{E_K + 2m}{(E_K + m)^2} f_{\bar{e}}(E_K)\right] \tag{B.30}$$

The A2c contribution

For the (A2c) contribution, first we use the expression for $\Lambda_{\lambda\rho}$ from Eq. (4.8) to find

$$\begin{aligned} \Lambda'_{\lambda\rho}(k,q) &= \eta_{\lambda\rho}(q^2 - 2k\cdot q)(\not{k} - 2m) + (k_\lambda q_\rho + k_\rho q_\lambda)(\not{k} - 4m) + k_\lambda \not{k}\gamma_\rho \not{q} + k_\rho \not{k}\gamma_\lambda \not{q}, \\ \Lambda''_{\lambda\rho}(k,q) &= \eta_{\lambda\rho}(q^2 + 2k\cdot q)(\not{k} - 2m) - (k_\lambda q_\rho + k_\rho q_\lambda)(\not{k} - 4m) - k_\lambda \not{q}\gamma_\rho \not{k} - k_\rho \not{q}\gamma_\lambda \not{k}, \end{aligned} \tag{B.31}$$

dropping irrelevant $O(q^2)$-terms and using $k^2 = m^2$. In the $\eta_{\lambda\rho}$ terms, the integrand becomes independent of q. Thus, these terms give a regular contribution. Let us denote it by (A2r):

$$m'_{(A2r)} \;=\; \frac{2e^2}{m}\int \frac{d^3K}{(2\pi)^3 2E_K}\left[\frac{E_K - 2m}{E_K - m} f_e(E_K) + \frac{E_K + 2m}{E_K + m} f_{\bar{e}}(E_K)\right]. \tag{B.32}$$

The terms which appear next will be called (A2s). For these, we use the fact that

$$(2v^\lambda v^\rho - \eta^{\lambda\rho})(k_\lambda q_\rho + k_\rho q_\lambda) = -2k \cdot q = 2\vec{K} \cdot \vec{Q}\,, \tag{B.33}$$

using $q \cdot v = q_0 = 0$. The $Q \to 0$ limit can then be taken easily, and we obtain

$$m'_{(A2s)} = -\frac{e^2}{m} \int \frac{d^3K}{(2\pi)^3 2E_K} \left[\frac{E_K - 4m}{E_K - m} f_e(E_K) + \frac{E_K + 4m}{E_K + m} f_{\bar{e}}(E_K) \right]. \tag{B.34}$$

Finally, we come to the terms with three γ-matrices, which we denote by (A2t). For these, first we note that

$$(2v^\lambda v^\rho - \eta^{\lambda\rho})(k_\lambda \not{k} \gamma_\rho \not{q} + k_\rho \not{k} \gamma_\lambda \not{q}) = 4k \cdot v \not{k} \not{v} \not{q} - 2m^2 \not{q}\,, \tag{B.35}$$

and a similar expression with the other term. Since the $\not{q}$ term vanishes between the spinors, we can write

$$\begin{aligned} m'_{(A2t)} &= 2e^2 \lim_{P\to 0} \lim_{Q \to 0} \int \frac{d^4k}{(2\pi)^3}\, \delta(k^2 - m^2) \eta_F(k) \\ &\quad \times \frac{k_0}{m^2 - k \cdot p}\, \frac{1}{2\vec{K} \cdot \vec{Q}}\, \overline{u}(p') \Big(\not{k} \not{v} \not{q} + \not{q} \not{v} \not{k} \Big) u(p)\,, \end{aligned} \tag{B.36}$$

omitting the Q^2 terms in the denominator since they will not contribute for $Q \to 0$. Using the identity

$$\gamma_\kappa \gamma_\mu \gamma_\nu = \eta_{\kappa\mu} \gamma_\nu + \eta_{\mu\nu} \gamma_\kappa - \eta_{\kappa\nu} \gamma_\mu - i \varepsilon_{\kappa\mu\nu\alpha} \gamma^\alpha \gamma_5\,, \tag{B.37}$$

we obtain

$$\not{k} \not{v} \not{q} + \not{q} \not{v} \not{k} = 2\vec{K} \cdot \vec{Q} \not{v} \tag{B.38}$$

between the spinors, since $q \cdot v = 0$ and $\not{q}$ terms vanish. Putting this back into Eq. (B.36) and using Eqs. (2.35) and (2.36), we obtain

$$m'_{(A2t)} = -\frac{2e^2}{m} \int \frac{d^3K}{(2\pi)^3 2E_K} \left[\frac{E_K}{E_K - m} f_e(E_K) + \frac{E_K}{E_K + m} f_{\bar{e}}(E_K) \right]. \tag{B.39}$$

The sum of Eqs. (B.29), (B.30), (B.32), (B.34) and (B.39) gives the total contribution of the A2 term, given in Eq. (5.12) in the text.

B.3 The X contribution

The part of the integral from Eq. (5.26) that we consider here is given by

$$I^{(f)}(Q) = \int \frac{d^3K}{(2\pi)^3} F(E_K) \left(\frac{1}{2\vec{K} \cdot \vec{Q} - Q^2} - \frac{1}{2\vec{K} \cdot \vec{Q} + Q^2} \right), \tag{B.40}$$

where

$$F(E) \equiv \Big(f_f(E) - f_{\bar{f}}(E) \Big) (2E^2 - m^2)\,. \tag{B.41}$$

Shifting the variables, the integral can be written as

$$I^{(f)}(Q) \quad = \quad \int \frac{d^3K}{(2\pi)^3} \frac{F(E_{\vec{K}+\frac{1}{2}\vec{Q}}) - F(E_{\vec{K}-\frac{1}{2}\vec{Q}})}{2\vec{K}\cdot\vec{Q}} . \tag{B.42}$$

We have to expand the numerator to $O(Q^3)$ in order to obtain the integral to $O(Q^2)$. Writing ∂_i to denote a partial derivative with respect to K^i,

$$F(E_{\vec{K}\pm\frac{1}{2}\vec{Q}}) = F(E) \pm \frac{1}{2}Q^i\partial_i F + \frac{1}{2}\left(\frac{1}{4}Q^iQ^j\right)\partial_i\partial_j F \pm \frac{1}{3!}\left(\frac{1}{8}Q^iQ^jQ^l\right)\partial_i\partial_j\partial_l F . \tag{B.43}$$

The derivatives we need to use are:

$$\begin{aligned} \partial_i F &= K^i\left(\frac{1}{E}\frac{\partial}{\partial E}\right)F , \\ \partial_i\partial_j\partial_l F &= (\delta^{ij}K^l + \delta^{il}K^j + \delta^{jl}K^i)\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^2 F + K^iK^jK^l\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^3 F . \end{aligned} \tag{B.44}$$

Using

$$K^iK^j \to \frac{1}{3}K^2\delta^{ij} \tag{B.45}$$

within the integrand, we have

$$I^{(f)}(Q) = \int \frac{d^3K}{(2\pi)^3}\left[\frac{1}{2}\left(\frac{1}{E}\frac{\partial}{\partial E}\right)F + \frac{1}{3!}\frac{Q^2}{8}\left\{3\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^2 F + \frac{1}{3}K^2\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^3 F\right\}\right] . \tag{B.46}$$

Therefore, the quantity that we must substitute in Eq. (5.23) is

$$I^{(f)}(Q) - I^{(f)}(Q\to 0) \quad = \quad \frac{1}{3!}\frac{Q^2}{8}\int \frac{d^3K}{(2\pi)^3}\left\{3\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^2 F + \frac{K^2}{3}\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^3 F\right\} . \tag{B.47}$$

We now use the identity

$$\int_0^\infty dK\, K^n\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^\nu F = -(n-1)\int_0^\infty dK\, K^{n-2}\left(\frac{1}{E}\frac{\partial}{\partial E}\right)^{\nu-1} F , \tag{B.48}$$

which holds for $n \geq 2$, so that the surface term vanishes. It is obtained by using Eq. (6.33) and performing a partial integration. Using it repeatedly, we can rewrite Eq. (B.47) as

$$I^{(f)}(Q) - I^{(f)}(Q\to 0) \quad = \quad -\frac{Q^2}{48\pi^2}\int_0^\infty dK\,\left(\frac{1}{E}\frac{\partial}{\partial E}\right)F . \tag{B.49}$$

Putting this back into Eq. (5.26), we obtain the total X contribution given in Eq. (5.27).

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